

On the (l, ω) -domination number of the cube network *

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Abstract For an n -connected graph G , the n -wide diameter $d_n(G)$, is the minimum integer m such that there are at least n internally disjoint (di)paths of length at most m between any vertices x and y . For a given integer l , a subset S of $V(G)$ is called an (l, n) - dominating set of G if for any vertex $x \in V(G) - S$ there are at least n internally disjoint (di)paths of length at most l from S to x . The minimum cardinality among all (l, n) -dominating sets of G is called the (l, n) -domination number. In this paper, we obtain that the (l, ω) -domination number of the d -ary cube network $C(d, n)$ is 2 for $1 \leq \omega \leq n$ and $d_\omega(G) - f(d, n) \leq l \leq d_\omega(G) - 1$ if $d, n \geq 4$, where $f(d, n) = \min\{e(\lfloor n/2 \rfloor + 1), \lceil n/2 \rceil e\}$.

Keywords: Cube network, Domination number, Wide diameter, Combinatorial problems

MR Subject Classification: 05C40 68M10 68M15 68R10

1 Introduction

This paper uses graphs to represent networks. The distance $d_G(x, y)$ from a vertex x to another vertex y in a network G is the minimum number of edges of a (di)path from x to y . The diameter $d(G)$ is the maximum distance from one vertex to another. The connectivity $k(G)$ is the minimum number of vertices whose removal results in a disconnected or trivial network.

In order to characterize the reliability of transmission delay in a real-time parallel processing system, Hsu and Lyuu [6], Flandrin and Li [4]

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independently introduced n -wide diameter. For an n -connected graph G , the distance with width n from x to y , denoted by $d_n(G; x, y)$, is the minimum number m for which there are n internally disjoint (x, y) -(di)paths in G of length at most m . The n -wide diameter of G , i.e., the n -diameter, denoted by $d_n(G)$, is the maximum of $d_n(G; x, y)$ over all pairs (x, y) of vertices of G .

Li and Xu [7] defined a new parameter (l, n) -domination number. This motives us to generalize the definition to that of the digraph. Let G be an n -connected digraph, S a nonempty and proper subset of $V(G)$, x a vertex in $G - S$. For a given positive integer l , x is (l, n) -dominated by S if there are at least n internally disjoint (S, x) -dipaths of length at most l . S is said to be an (l, n) -dominating set of G if S can (l, n) -dominate any vertex in $G - S$. The minimum cardinality among all (l, n) -dominating sets of G is called the (l, n) -domination number, denoted by $\gamma_{l,n}(G)$.

The d -ary cube network $C(d, n)$ is a digraph of d^n vertices, in which any vertex x has the form $(x_{n-1}, x_{n-2}, \dots, x_0)$ where $0 \leq x_i \leq d-1$ for $0 \leq i \leq n-1$, and x is adjacent to $(x_{n-1}, \dots, x_{j+1}, x_j + 1, x_{j-1}, \dots, x_0)$ for $0 \leq j \leq n-1$, where additions are taken modulo d . $C(2, n)$ is the n -dimensional binary hypercube Q_n . It is clear that $C(d, n)$ is vertex-transitive and its diameter is $n(d-1)$. Hsu and Lyuu [6] proved that $d_n(C(d, n)) = n(d-1) + 1$. Liaw and Chang [8] showed that $d_\omega(C(d, n)) = n(d-1)$ for $1 \leq \omega \leq n-1$ and $d_n(C(d, n)) = n(d-1) + 1$. Since $\gamma_{l,n}(G) = 1$ for $l \geq d_n(G)$ and $\gamma_{l,n}(G) \geq 2$ for $l < d_n(G)$, so it is of interest to show some general properties and values of the (l, ω) -domination numbers of n -connected graphs for $l < d_n(G)$ and $1 \leq \omega \leq n$ (see, for example [1, 2, 5, 7, 9, 10, 11, 12]).

In this paper, we obtain $\gamma_{l,\omega}(C(d, n)) = 2$ for $1 \leq \omega \leq n$ and $d_\omega(G) - f(d, n) \leq l \leq d_\omega(G) - 1$ if $d, n \geq 4$, where $f(d, n) = \min\{e(\lfloor n/2 \rfloor + 1), \lceil n/2 \rceil e'\}$.

Terminologies and notations not defined here are referred to [3].

2 Preliminaries

Let $c_i(x) = x_i$ denote the i th component of vertex $x = (x_{n-1}, x_{n-2}, \dots, x_0)$. For $0 \leq i \leq n-1$, the i th unit vector is the vector e_i^n with $c_i(e_i^n) = 1$ and $c_j(e_i^n) = 0$ for $0 \leq j \leq n-1$, with $j \neq i$. The vertex set of $C(d, n)$ can be viewed as a module over Z_d . So vertex x can also be written as $x = \sum_{i=0}^{n-1} x_i e_i^n$. Denote $\lfloor d/2 \rfloor$ and $\lceil d/2 \rceil$ by e and e' , respectively. Let $f(d, n) = \min\{e(\lfloor n/2 \rfloor + 1), \lceil n/2 \rceil e'\}$ in this paper.

Suppose a_0, a_1, \dots, a_r are positive integers and $0 \leq i_0 \leq i_1 \leq \dots \leq i_r \leq n-1$ for $0 \leq r \leq n-1$. Denote by $\langle\langle a_0 e_{i_0}^n(o), a_1 e_{i_1}^n(o), \dots, a_r e_{i_r}^n(o) \rangle\rangle$

the following dipath from vertex $o = (0, 0, \dots, 0)$ to vertex $\sum_{j=0}^r a_j e_{i_j}^n$:

$$\begin{aligned} o &\rightarrow e_{i_0}^n \rightarrow 2e_{i_0}^n \rightarrow \dots \rightarrow a_0 e_{i_0}^n \\ &\rightarrow a_0 e_{i_0}^n + e_{i_1}^n \rightarrow a_0 e_{i_0}^n + 2e_{i_1}^n \rightarrow \dots \rightarrow a_0 e_{i_0}^n + a_1 e_{i_1}^n \\ &\rightarrow \dots \\ &\rightarrow \sum_{j=0}^{r-1} a_j e_{i_j}^n + e_{i_r}^n \rightarrow \sum_{j=0}^{r-1} a_j e_{i_j}^n + 2e_{i_r}^n \rightarrow \dots \rightarrow \sum_{j=0}^r a_j e_{i_j}^n, \end{aligned}$$

and by $\ll a_0 e_{i_0}^n(v), a_1 e_{i_1}^n(v), \dots, a_r e_{i_r}^n(v) \gg$ the following dipath from vertex $v = (e, e, \dots, e)$ to vertex $v + \sum_{j=0}^r a_j e_{i_j}^n$:

$$\begin{aligned} v &\rightarrow v + e_{i_0}^n \rightarrow v + 2e_{i_0}^n \rightarrow \dots \rightarrow v + a_0 e_{i_0}^n \\ &\rightarrow v + a_0 e_{i_0}^n + e_{i_1}^n \rightarrow v + a_0 e_{i_0}^n + 2e_{i_1}^n \rightarrow \dots \rightarrow v + a_0 e_{i_0}^n + a_1 e_{i_1}^n \\ &\rightarrow \dots \\ &\rightarrow v + \sum_{j=0}^{r-1} a_j e_{i_j}^n + e_{i_r}^n \rightarrow v + \sum_{j=0}^{r-1} a_j e_{i_j}^n + 2e_{i_r}^n \rightarrow \dots \rightarrow v + \sum_{j=0}^r a_j e_{i_j}^n. \end{aligned}$$

3 Main results

Lemma 3.1 Let $S = \{o, v\}$ be a subset of $V(C(d, n))$ with $o = (0, 0, \dots, 0)$ and $v = (e, e, \dots, e)$, $d, n \geq 4$. Then there exists n internally disjoint dipaths of length at most $n(d-1) - f(n, d) + 1$ from S to $x \in V(C(d, n)) - S$ if vertex x has no zero components.

Proof Since the digraph is vertex-transitive, without loss of generality, we consider the following cases for any vertex x with no zero components in $V(C(d, n)) - S$:

Case 1. Vertex x has no components with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_j}^{n-j}, \overbrace{x_{j-1}, \dots, x_0}^j)$ for $e+1 \leq x_{n-1}, \dots, x_j \leq d-1$ and $1 \leq x_{j-1}, \dots, x_0 \leq e-1$.

Subcase 1a. $\lceil n/2 \rceil \leq j \leq n$. Construct n internally disjoint dipaths from o to x as follows:

$$P_t : \ll x_t e_t^n(o), x_{t+1} e_{t+1}^n(o), \dots, x_{n-1} e_{n-1}^n(o), x_0 e_0^n(o), x_1 e_1^n(o), \dots, x_{t-1} e_{t-1}^n(o) \gg \quad \text{for } 0 \leq t \leq n-1.$$

We can see that the length of each dipath is

$$\sum_{l=0}^{n-1} x_l \leq j(e-1) + (n-j)(d-1) = n(d-1) - j e' \leq n(d-1) - \lceil n/2 \rceil e'.$$

Subcase 1b. $0 \leq j \leq \lceil n/2 \rceil - 1$. By vertex-transitive, we can construct n internally disjoint dipaths from v to x in the same way as in Subcase 1a, and the length of each dipath is

$$\begin{aligned}
\sum_{l=0}^{j-1} (e' + x_l) + \sum_{l=j}^{n-1} (x_l - e) &\leq j(d-1) + (n-j)(e' - 1) \\
&= n(e' - 1) + je \\
&\leq n(e' - 1) + e(\lceil n/2 \rceil - 1) \\
&= n(d-1) - e(\lfloor n/2 \rfloor + 1).
\end{aligned}$$

Case 2. Vertex x has some components with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_{j+k}}^{n-j-k}, \overbrace{e, \dots, e}^k, \overbrace{x_{j-1}, \dots, x_0}^j)$ for $e+1 \leq x_{n-1}, \dots, x_{j+k} \leq d-1$ and $1 \leq x_{j-1}, \dots, x_0 \leq e-1$, $k \geq 1$.

Subcase 2a. $\lceil n/2 \rceil - 1 \leq j \leq n$. Construct the same dipaths from o to x as in Subcase 1a. The length of each dipath is

$$\begin{aligned}
\sum_{l=0}^{j-1} x_l + ke + \sum_{l=j+k}^{n-1} x_l &\leq j(e-1) + ke + (n-j-k)(d-1) \\
&= n(d-1) - je' - k(e' - 1) \\
&\leq n(d-1) - (\lceil n/2 \rceil - 1)e' - (e' - 1) \\
&= n(d-1) - \lceil n/2 \rceil e' + 1.
\end{aligned}$$

Subcase 2b. $0 \leq j \leq \lceil n/2 \rceil - 2$. Construct n internally disjoint dipaths from v to x as follows:

$P_t : << (e' + x_t)e_t^n(v), (e' + x_{t+1})e_{t+1}^n(v), \dots, (e' + x_{j-1})e_{j-1}^n(v), (x_{j+k} - e)e_{j+k}^n(v), (x_{j+k+1} - e)e_{j+k+1}^n(v), \dots, (x_{n-1} - e)e_{n-1}^n(v), (e' + x_0)e_0^n(v), (e' + x_1)e_1^n(v), \dots, (e' + x_{t-1})e_{t-1}^n(v) >>$ for $0 \leq t \leq j-1$;

$P_t : << (d-1)e_t^n(v), (x_{j+k} - e)e_{j+k}^n(v), (x_{j+k+1} - e)e_{j+k+1}^n(v), \dots, (x_{n-1} - e)e_{n-1}^n(v), (e' + x_0)e_0^n(v), (e' + x_1)e_1^n(v), \dots, (e' + x_{j-1})e_{j-1}^n(v), e_t^n(v) >>$ for $j \leq t \leq j+k-1$;

$P_t : << (x_t - e)e_t^n(v), (x_{t+1} - e)e_{t+1}^n(v), \dots, (x_{n-1} - e)e_{n-1}^n(v), (e' + x_0)e_0^n(v), (e' + x_1)e_1^n(v), \dots, (e' + x_{j-1})e_{j-1}^n(v), (x_{j+k} - e)e_{j+k}^n(v), (x_{j+k+1} - e)e_{j+k+1}^n(v), \dots, (x_{t-1} - e)e_{t-1}^n(v) >>$ for $j+k \leq t \leq n-1$.

The length of each dipath is at most

$$\begin{aligned}
&\sum_{l=0}^{j-1} (e' + x_l) + \sum_{l=j+k}^{n-1} (x_l - e) + d \\
&\leq j(d-1) + (n-j-k)(e' - 1) + d \\
&= n(e' - 1) + je - k(e' - 1) + d \\
&\leq n(e' - 1) + (\lceil n/2 \rceil - 2)e - e' + 1 + d \\
&= n(d-1) - e(\lfloor n/2 \rfloor + 1) + 1.
\end{aligned}$$

Summarizing cases 1 and 2, the length of each dipath is at most $n(d-1) - f(n, d) + 1$. ■

Lemma 3.2 Let $S = \{o, v\}$ be a subset of $V(C(d, n))$ with $o = (0, 0, \dots, 0)$ and $v = (e, e, \dots, e)$, $d, n \geq 4$. Then there exists n internally disjoint dipaths of length at most $n(d-1) - f(n, d) + 1$ from S to $x \in V(C(d, n)) - S$ if vertex x has some zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_{i+j}}^{n-i-j}, \overbrace{x_{i+j-1}, \dots, x_i}^j, \overbrace{0, \dots, 0}^i)$ for $e+1 \leq x_{n-1}, \dots, x_{i+j} \leq d-1$ and $1 \leq x_{i+j-1}, \dots, x_i \leq e-1$, $i \geq 1$.

Subcase 1a. $\lceil n/2 \rceil + 1 \leq i+j \leq n$. Construct n internally disjoint dipaths from o to x in the same way as in Subcase 2b of Lemma 3.1. The length of each dipath is at most

$$\begin{aligned} \sum_{l=i}^{n-1} x_l + d &\leq j(e-1) + (n-i-j)(d-1) + d \\ &= n(d-1) - i(d-1) - je' + d \\ &\leq n(d-1) - i(d-1) - (\lceil n/2 \rceil + 1 - i)e' + d \\ &= n(d-1) - i(e-1) - (\lceil n/2 \rceil + 1)e' + d \\ &\leq n(d-1) - (e-1) - (\lceil n/2 \rceil + 1)e' + d \\ &\leq n(d-1) - \lceil n/2 \rceil e' + 1. \end{aligned}$$

Subcase 1b. $0 \leq i+j \leq \lceil n/2 \rceil$. Construct the same dipaths from v to x as in Subcase 1b of Lemma 3.1. The length of each dipath P_t is

$$\begin{aligned} ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \\ \leq ie' + j(d-1) + (n-i-j)(e'-1) \\ = n(e'-1) + i + je \\ \leq n(e'-1) + i + (\lceil n/2 \rceil - i)e \\ = n(e'-1) - i(e-1) + \lceil n/2 \rceil e \\ \leq n(e'-1) - (e-1) + \lceil n/2 \rceil e \\ = n(d-1) - e(\lceil n/2 \rceil + 1) + 1. \end{aligned}$$

Case 2. Vertex has some component with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_{i+j+k}}^{n-i-j-k}, \overbrace{e, \dots, e}^k, \overbrace{x_{i+j-1}, \dots, x_i}^j, \overbrace{0, \dots, 0}^i)$ for $e+1 \leq x_{n-1}, \dots, x_{i+j+k} \leq d-1$ and $1 \leq x_{i+j-1}, \dots, x_i \leq e-1$, $i, k \geq 1$.

Subcase 2a. $\lceil n/2 \rceil + 1 \leq i+j \leq n$. Construct n internally disjoint dipaths from o to x in the same way as in Subcase 2b of Lemma 3.1. So

the length of each dipath is at most

$$\begin{aligned}
\sum_{l=i}^{n-1} x_l + d &\leq j(e-1) + ke + (n-i-j-k)(d-1) + d \\
&= n(d-1) - i(d-1) - je' - k(e'-1) + d \\
&\leq n(d-1) - i(d-1) - (\lceil n/2 \rceil + 1 - i)e' - k(e'-1) + d \\
&= n(d-1) - i(e-1) - (\lceil n/2 \rceil + 1)e' - k(e'-1) + d \\
&\leq n(d-1) - (e-1) - (\lceil n/2 \rceil + 1)e' - (e'-1) + d \\
&= n(d-1) - e'(\lceil n/2 \rceil + 1) + 2.
\end{aligned}$$

Subcase 2b. $i+j = \lceil n/2 \rceil$. Construct $n-i-j$ internally disjoint dipaths from o to x and $i+j$ internally disjoint dipaths from v to x as follows:

$P_t : << x_i e_i^n(o), x_{i+1} e_{i+1}^n(o), \dots, x_{i+j-1} e_{i+j-1}^n(o), x_t e_t^n(o), x_{t+1} e_{t+1}^n(o), \dots, x_{n-1} e_{n-1}^n(o), x_{i+j} e_{i+j}^n(o), x_{i+j+1} e_{i+j+1}^n(o), \dots, x_{t-1} e_{t-1}^n(o) >>$ for $i+j \leq t \leq n-1$;

$P_t : << (e' + x_t) e_t^n(v), (e' + x_{t+1}) e_{t+1}^n(v), \dots, (e' + x_{i+j-1}) e_{i+j-1}^n(v), (x_{i+j+k} - e) e_{i+j+k}^n(v), (x_{i+j+k+1} - e) e_{i+j+k+1}^n(v), \dots, (x_{n-1} - e) e_{n-1}^n(v), (e' + x_0) e_0^n(v), (e' + x_1) e_1^n(v), \dots, (e' + x_{t-1}) e_{t-1}^n(v) >>$ for $0 \leq t \leq i+j-1$.

The length of dipath P_t for $i+j \leq t \leq n-1$ is $\sum_{l=i}^{n-1} x_l$ and the length of dipath P_t for $0 \leq t \leq i+j-1$ is $\sum_{l=0}^{i+j-1} (e' + x_l) + \sum_{l=i+j+k}^{n-1} (x_l - e)$. Noting $\sum_{l=i}^{n-1} x_l \leq \sum_{l=0}^{i+j-1} (e' + x_l) + \sum_{l=i+j+k}^{n-1} (x_l - e)$ for $i+j = \lceil n/2 \rceil$, so the length of each dipath P_t is at most

$$\begin{aligned}
&\sum_{l=0}^{i+j-1} (e' + x_l) + \sum_{l=i+j+k}^{n-1} (x_l - e) \\
&\leq ie' + j(d-1) + (n-i-j-k)(e'-1) \\
&= n(e'-1) + i + je - k(e'-1) \\
&= n(e'-1) + i + (\lceil n/2 \rceil - i)e - k(e'-1) \\
&= n(e'-1) + \lceil n/2 \rceil e - i(e-1) - k(e'-1) \\
&\leq n(e'-1) + \lceil n/2 \rceil e - (e-1) - (e'-1) \\
&= n(d-1) - \lfloor n/2 \rfloor e - d + 2.
\end{aligned}$$

Subcase 2c. $i+j = \lceil n/2 \rceil - 1$. Construct the same dipaths as in Subcase 2b of Lemma 3.2. Similarly, $\sum_{l=0}^{i+j-1} (e' + x_l) + \sum_{l=i+j+k}^{n-1} (x_l - e) \leq n(d-1) - \lfloor n/2 \rfloor e - d - e + 2$, and

$$\begin{aligned}
\sum_{l=i}^{n-1} x_l &\leq j(e-1) + ke + (n-i-j-k)(d-1) \\
&= n(d-1) - i(d-1) - je' - k(e'-1) \\
&= n(d-1) - i(d-1) - (\lceil n/2 \rceil - 1 - i)e' - k(e'-1) \\
&= n(d-1) - i(e-1) - (\lceil n/2 \rceil - 1)e' - k(e'-1) \\
&\leq n(d-1) - (e-1) - (\lceil n/2 \rceil - 1)e' - (e'-1) \\
&= n(d-1) - e - \lceil n/2 \rceil e' + 2.
\end{aligned}$$

Subcase 2d. $0 \leq i+j \leq \lceil n/2 \rceil - 2$. Construct the same dipaths as in Subcase 2b of Lemma 3.1. So the length of dipath is at most

$$\begin{aligned}
ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j+k}^{n-1} (x_l - e) + d \\
\leq ie' + j(d-1) + (n-i-j-k)(e'-1) + d \\
= n(e'-1) + i + je - k(e'-1) + d \\
\leq n(e'-1) + i + (\lceil n/2 \rceil - 2 - i)e - k(e'-1) + d \\
= n(e'-1) - i(e-1) + (\lceil n/2 \rceil - 2)e - k(e'-1) + d \\
\leq n(e'-1) - (e-1) + (\lceil n/2 \rceil - 2)e - (e'-1) + d \\
= n(d-1) - e(\lfloor n/2 \rfloor + 2) + 2.
\end{aligned}$$

Summarizing cases 1 and 2, the length of each dipath is at most $n(d-1) - f(n, d) + 1$. ■

Finally, we can see that Lemma 3.1 and 3.2 yield the following theorem.

Theorem 3.3 If $d, n \geq 4$, then $\gamma_{l,n}(C(d, n)) = 2$ for $n(d-1) - f(n, d) + 1 \leq l \leq n(d-1)$. ■

Lemma 3.4 Let $S = \{o, v\}$ be a subset of $V(C(d, n))$ with $o = (0, 0, \dots, 0)$ and $v = (e, e, \dots, e)$, $d, n \geq 4$. For $1 \leq \omega \leq n-1$, there exists ω internally disjoint dipaths of length at most $n(d-1) - f(n, d)$ from S to $x \in V(C(d, n)) - S$ if vertex x has no zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e .

From the Case 1 of Lemma 3.1, the result follows.

Case 2. Vertex x has some component with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_{j+k}}^{n-j-k}, \overbrace{e, \dots, e}^k, \overbrace{x_{j-1}, \dots, x_0}^j)$ for $e+1 \leq x_{n-1}, \dots, x_{j+k} \leq d-1$ and $1 \leq x_{j-1}, \dots, x_0 \leq e-1$, $k \geq 1$.

Subcase 2a. $\lceil n/2 \rceil \leq j \leq n$. Construct the same ω internally disjoint dipaths as in Subcase 1a of Lemma 3.1, and we can easily see the length of

each dipath is

$$\sum_{l=0}^{j-1} x_l + ke + \sum_{l=j+k}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - e'.$$

The details are omitted here.

Subcase 2b. $0 \leq j \leq \lceil n/2 \rceil - 1$.

If $k = 1$, construct the same ω internally disjoint dipaths as P_t for $0 \leq t \leq j-1$ and $j+1 \leq t \leq n-1$ in Subcase 2b of Lemma 3.1. Similarly, the length of each dipath is

$$\sum_{l=0}^{j-1} (e' + x_l) + \sum_{l=j+1}^{n-1} (x_l - e) \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 1 - e'.$$

Otherwise, $k \geq 2$. We consider the following cases:

For $j = \lceil n/2 \rceil - 1$, the desired ω internally disjoint dipaths are similar to that in Subcase 1a of Lemma 3.1, the length of each dipath is

$$\sum_{l=0}^{j-1} x_l + ke + \sum_{l=j+k}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 2 - e'.$$

For $j \leq \lceil n/2 \rceil - 2$, the desired ω internally disjoint dipaths are similar to that in Subcase 2b of Lemma 3.1, the length of each dipath is at most

$$\sum_{l=0}^{j-1} (e' + x_l) + \sum_{l=j+1}^{n-1} (x_l - e) + d \leq n(d-1) - e(\lfloor n/2 \rfloor + 1) + 2 - e'.$$

■

Lemma 3.5 Let $S = \{o, v\}$ be a subset of $V(C(d, n))$ with $o = (0, 0, \dots, 0)$ and $v = (e, e, \dots, e)$, $d, n \geq 4$. For $1 \leq \omega \leq n-1$, there exists ω internally disjoint dipaths of length at most $n(d-1) - f(n, d)$ from S to $x \in V(C(d, n)) - S$ if vertex x has some zero components.

Proof We consider the following cases:

Case 1. Vertex x has no components with value e .

Assume $x = (\overbrace{x_{n-1}, \dots, x_{i+j}}^{n-i-j}, \overbrace{x_{i+j-1}, \dots, x_i}^j, \overbrace{0, \dots, 0}^i)$ for $e+1 \leq x_{n-1}, \dots, x_{i+j} \leq d-1$ and $1 \leq x_{i+j-1}, \dots, x_i \leq e-1$, $i \geq 1$.

Subcase 1a. $\lceil n/2 \rceil + 1 \leq i+j \leq n$. We can construct ω internally disjoint dipaths from o to x .

If $i = 1$, the length of each dipath is

$$\sum_{l=i}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - d.$$

If $i \geq 2$, the length of each dipath is at most

$$\sum_{l=i}^{n-1} x_l + d \leq n(d-1) - \lceil n/2 \rceil e' + 2 - e.$$

Subcase 1b. $i + j = \lceil n/2 \rceil$. If $i \geq 2$, construct ω internally disjoint dipaths from v to x , the length of each dipath is

$$ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \leq n(d-1) - e(\lceil n/2 \rceil + 1) + 2 - e.$$

If $i = 1$, it is similar to the case of $i = 1$ in Subcase 1a of Lemma 3.5. The length of each dipath is

$$\sum_{l=i}^{n-1} x_l \leq n(d-1) - \lceil n/2 \rceil e' + 1 - e.$$

Subcase 1c. $i + j \leq \lceil n/2 \rceil - 1$. Construct ω internally disjoint dipaths from v to x , the length of each dipath is

$$ie' + \sum_{l=i}^{i+j-1} (e' + x_l) + \sum_{l=i+j}^{n-1} (x_l - e) \leq n(d-1) - e(\lceil n/2 \rceil + 1) + 1 - e.$$

Case 2. Vertex x has some component with value e .

From the Case 2 of Lemma 3.2, the result follows. ■

Finally, we can see that Lemma 3.4 and 3.5 yield the following theorem.

Theorem 3.6 If $d, n \geq 4$, then $\gamma_{l,\omega}(C(d, n)) = 2$ for $1 \leq \omega \leq n - 1$ and $n(d-1) - f(n, d) \leq l \leq n(d-1) - 1$. ■

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