113107 001-448 int NB_Ok-Proofs_PG 193_2015-03-16_16:49:59_K

On the E_k-cordiality of some graphs

Qing Liu *, Zhishan Liu †, Haiping Wu‡
wwhhpp88@163.com

Abstract

In this paper, we first provide two necessary conditions for a graph G to be E_k-cordial, then we prove that every $P_n(n \geq 3)$ is E_p-cordial if p is odd. In the end, we discuss the E_2-cordiality of a graph G under the condition that some subgraph of G has a 1-factor.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For an integer k, an edge label $f : E \to \{1, 2, \ldots, k-1\}$ induces the vertex label $f : V \to \{1, 2, \ldots, k-1\}$ defined by $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } k)$, where $N(v)$ is the set of vertices adjacent to v and $f(v) \in \{0, 1, \ldots, k-1\}$. The number of vertices(resp. edges) of G labelled with i under f will be denoted by $v_f(i, G)$(resp. $e_f(i, G)$), $i = 0, 1, \ldots, k-1$. A label f of G is said to be E_k-cordial if $|v_f(i, G) - v_f(j, G)| \leq 1$ and $|e_f(i, G) - e_f(j, G)| \leq 1, \forall i, j \in \{0, 1, 2, \ldots, k-1\}$.

Definition. A graph G is said to be E_k-cordial if it admits an E_k-cordial label.

The notion of an E_k-cordial label was first introduced by Cahit and Yilmaz[1], who showed that the following graphs are E_3-cordial: $P_n(n \geq 3)$, Stars S_n if and only if $n \not\equiv 1(\text{mod}3)$, $K_n(n \geq 3)$, $C_n(n \geq 3)$, friendship graphs and fans$(n \geq 3)$. They also proved $S_n(n \geq 2)$ is E_k-cordial if and only if $n \not\equiv 1(\text{mod } k)$ when k is odd or $n \not\equiv 1(\text{mod}2k)$ when k is even and $k \neq 2$. Some further results see [2]-[3].

*School of Statistics and Research Center of Applied Statistics, Jiangxi University of Finance and Economics, Nanchang 330013, P.R.China.

†Department of Mathematics, Yang-en University, Quanzhou 362014, P.R.China.

‡Correspondence author. School of Tourism and Urban Management, Jiangxi University of Finance and Economics, Nanchang 330013, P.R.China.

In this paper, we first provide two necessary conditions for a graph G to be E_k-cordial, then we prove that every $P_n (n \geq 3)$ is E_p-cordial if p is odd. In the end, we discuss the E_2-cordiality of a graph G under the condition that some subgraph of G has a 1-factor.

2 The E_k-Cordiality

We first present two necessary conditions for a graph to be E_k-cordial.

Theorem 1. If G is a graph of order $(2r+1)(4k+2)$, then G is not E_{4k+2}-cordial.

Proof. Suppose f is a label of G. Since every edge is incident with two vertices, so we have $\sum\limits_{v \in V(G)} f(v) \equiv 2 \sum\limits_{e \in E(G)} f(e) (\text{mod } 4k+2)$, which implies that $\sum\limits_{v \in V(G)} f(v)$ is even. Denote by v_{od} the number of vertices labelled with odd number, then v_{od} is even. However, if f is E_{4k+2}-cordial, then $v_f(i; G) = 2r+1, i \in \{0, 1, \ldots, 4k+1\}$. Hence $v_{od} = (2k+1)(2r+1)$, a contradiction.

Theorem 2. If G is a graph of order $(2r+1)4k$ and $e(G) = 4km$, then G is not E_{4k}-cordial.

Proof. Suppose f is an E_{4k}-cordial label of G. Then $v_f(i; G) = 2r+1$ and $e_f(i; G) = m$, $i \in \{0, 1, \ldots, 4k-1\}$. So $\sum\limits_{v \in V(G)} f(v) = (2r+1)(0+1+\ldots+4k-1) = 2k(2r+1)(4k-1)$, which indicates that $4k \nmid \sum\limits_{v \in V(G)} f(v)$. However, as $\sum\limits_{v \in V(G)} f(v) \equiv 2 \sum\limits_{e \in E(G)} f(e) (\text{mod } 4k)$ and $\sum\limits_{e \in E(G)} f(e) = m(0+1+\ldots+4k-1) = 2km(4k-1)$, then $\sum\limits_{v \in V(G)} f(v) \equiv 4km(4k-1)$. Hence $4k \mid \sum\limits_{v \in V(G)} f(v)$, a contradiction.

In the end of this section, we consider the E_p-cordiality of $P_n (n \geq 3)$.

Theorem 3. If p is odd, then $P_n (n \geq 3)$ is E_p-cordial.

Proof. In order to give the E_p-cordial label of P_n, we distinguish four cases.

Case 1. If $n \leq p$ and n is odd, the edges of P_n are labelled in order with $1, 2, \ldots, n-1$.

Case 2. If $n \leq p$ and n is even, we first label the edges of P_n in order with even numbers and then with odd numbers from 5, i.e. the edges of P_n

are labelled with $0, 2, 4, \ldots, p-3, p-1, 5, 7, \ldots, p-2, p$.

Case 3. If $n = kp+1$ $(k \geq 1)$, the edges of P_n are labelled in order with $\underbrace{0, 0, \ldots, 0}_{k}$, $\underbrace{1, 1, \ldots, 1}_{k}$, $\underbrace{2, 2, \ldots, 2}_{k}$, $\ldots$, $\underbrace{p-1, p-1, \ldots, p-1}_{k}$.

Case 4. If $n = kp + m + 1$ $(k \geq 1, 1 \leq m \leq p-1)$, the edges of P_n are labelled in order with $\underbrace{0, 0, \ldots, 0}_{k}$, $\underbrace{1, 1, \ldots, 1}_{k+1}$, $\underbrace{2, 2, \ldots, 2}_{k+1}$, $\ldots$, $\underbrace{m, m, \ldots, m}_{k+1}$, $\underbrace{m+1, m+1, \ldots, m+1}_{k}$, $\ldots$, $\underbrace{p-1, p-1, \ldots, p-1}_{k}$.

In any case, we can check that the label given above is E_p-cordial. Therefore, $P_n (n \geq 3)$ is E_p-cordial when p is odd.

3 The E_2-Cordiality

From the conclusion of Theorem 1, we can see that a graph of order $4k+2$ is not E_2-cordial. In this section, we discuss the E_2-cordiality of a graph G when G is not of order $4k+2$ and some subgraph of it has a 1-factor.

Theorem 4. If a graph G of order $4n$ has a 1-factor, i.e., $V(G) = \{x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}\}$ and $\{x_1y_1, x_2y_2, \ldots, x_{2n}y_{2n}\} \subseteq E(G)$, then G is E_2-cordial.

Proof. Let H be a subgraph of G such that $E(H) = E(G)/\{x_1y_1, x_2y_2, \ldots, x_{2n}y_{2n}\}$ and $V(H) = V(G)$. We first define the edge label of H by $f : E(H) \to \{0, 1\}$ and $|e_f(1, H) - e_f(0, H)| \leq 1$, then we induce the vertex label of H by $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } 2)$. According to the label of end-points under f, the $2n$ edges $x_1y_1, x_2y_2, \ldots, x_{2n}y_{2n}$ fall into three groups. Without loss of generality, we write $E_{11} = \{x_1y_1, x_2y_2, \ldots, x_ky_k\}$, $E_{00} = \{x_{k+1}y_{k+1}, \ldots, x_{k+m}y_{k+m}\}$ and $E_{01} = \{x_{k+m+1}y_{k+m+1}, \ldots, x_{2n}y_{2n}\}$, where $f(x_i) = f(y_i) = \begin{cases} 1, & i = 1, 2, \ldots, k \\ 0, & i = k+1, \ldots, k+m \end{cases}$ and $f(x_i) \neq f(y_i), i = k + m + 1, \ldots, 2n$.

In order to find out the E_2-cordial label of G we need further to label the rest $2n$ edges $x_1y_1, x_2y_2, \ldots, x_{2n}y_{2n}$ properly. Suppose $t = 2n - k - m$. From the fact that $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } 2)$, we can conclude that t is even and k and m are of the same parity. We consider the following cases.

Case 1. $t \geq 2$. If k and m are both even, we label one half of edges in E_{01} with 0 and the other with 1. Likewise, $\frac{k}{2}$ edges in E_{11} and $\frac{m}{2}$ in E_{00} are labelled with 0 and the rest edges in E_{11} and E_{00} are labelled with 1. Besides, the edges of H are labelled with f. Then we obtain an E_2-cordial label of G. Similarly, when k and m are both odd, $\frac{k+1}{2}$ edges in E_{11}, $\frac{m+1}{2}$ in E_{00} and $\frac{t-2}{2}$ in E_{01} are labelled with 0 and others with 1.

Case 2. $t = 0$. If k and m are both even, $\frac{k}{2}$ edges in E_{11} and $\frac{m}{2}$ in E_{00} are labelled with 0 and others with 1. If k and m are both odd, since $f(x_1) = 1$, there exists a vertex $x \in V(G)$ such that $xx_1 \in E(H)$ and $f(xx_1) = 1$.

Subcase 1. $f(x) = 0$. Without loss of generality, we may assume $x = x_{k+1}$. We define a label g of G as follows. If $e \in E(H)/\{x_1x_{k+1}\}$, let $g(e) = f(e)$; $g(x_1x_{k+1}) = 0$; $g(x_1y_1) = g(x_{k+1}y_{k+1}) = 1$. Besides, $\frac{k-1}{2}$ edges in $E_{11}/\{x_1y_1\}$ and $\frac{m-1}{2}$ in $E_{00}/\{x_{k+1}y_{k+1}\}$ are labelled with 0 and others with 1. Hence g is E_2-cordial.

Subcase 2. $f(x) = 1$. Without loss of generality, we may assume $x = x_2$. We can find an E_2-cordial label g of G as follows. If $e \in E(H)/\{x_1x_2\}$, let $g(e) = f(e)$; $g(x_1x_2) = 0$; $g(x_1y_1) \neq g(x_2y_2)$. Besides, $\frac{k-1}{2}$ edges in $E_{11}/\{x_1y_1, x_2y_2\}$ and $\frac{m+1}{2}$ in E_{00} are labelled with 1 and others with 0. It is easy to check that g is an E_2-cordial label of G.

Denote by $|G|$ the order of a graph G. By using Theorem 4, we can obtain the following corollaries easily.

Corollary 1. Suppose a graph G has a perfect matching, then the Cartesian product $G \times P_2$ is E_2-cordial if and only if $|G| = 2n$.

Corollary 2. Suppose G and H are two graphs and G has a perfect matching, then the Cartesian product $G \times H$ is E_2-cordial if and only if $|G| \cdot |H| = 4n$.

Theorem 5. Suppose G is a graph of order $4n+1$. If $V(G) = \{x_1, \dots, x_{2n}, y_1, \dots, y_{2n}, w\}$ and $\{x_1y_1, x_2y_2, \dots, x_{2n}y_{2n}\} \subseteq E(G)$, then G is E_2-cordial.

Proof. Let M be a subgraph of G such that $E(M) = E(G)/\{x_1y_1, x_2y_2, \dots, x_{2n}y_{2n}\}$ and $V(M) = V(G)$. We first label $E(M)$ with 0 and 1 such that the difference between the number of edges labelled with 0 and that labelled with 1 is at most 1. Then we derive the vertex label of M by $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } 2)$ and $f(v) \in \{0, 1\}$. Denote by f the label of M defined above and E_{00}, E_{11}, E_{01} the same meaning as in Theorem 4. In order to label the rest edges $x_1y_1, x_2y_2, \dots, x_{2n}y_{2n}$, we distinguish two cases.

Case 1. $t = 2n - k - m > 0$.

Subcase 1. t is even. As the number of vertices labelled with 1 is even, we can see that $f(w) = 0$. So k and m are of the same parity. Label the $2n$ edges as in Case 1 of Theorem 4 and we get an E_2-cordial label of G.

Subcase 2. t is odd. It follows that $f(w) = 1$ and k and m are of different parity. If k is odd, $\frac{k-1}{2}$ edges in E_{11}, $\frac{m}{2}$ in E_{00} and $\frac{t+1}{2}$ in E_{01} are labelled with 0 and others with 1. The result is an E_2-cordial label of G. Conversely, if m is odd, $\frac{k}{2}$ in E_{11}, $\frac{m+1}{2}$ in E_{00} and $\frac{t-1}{2}$ in E_{01} are labelled with 0 and others with 1.

Case 2. $t = 0$.

It is obvious that $f(w) = 0$ and k and m are of the same parity. If k and m are both even, $\frac{k}{2}$ edges in E_{11} and $\frac{m}{2}$ in E_{00} are labelled with 0 and others with 1. If k and m are both odd, since $f(x_1) = f(y_1) = 1$, there exist two vertices $x, y \in V(M)$ such that $f(xx_1) = f(yy_1) = 1$. If $x \neq w$ or $y \neq w$, then G is E_2-cordial by the same discussion as Case 2 of Theorem 4. If $x = y = w$, in term of the definition of f, there must be an edge $e_0 \in E(M)$ such that $f(e_0) = 0$. By changing the label of x_1w with that of e_0, we get another label $f^{'}$ of M satisfying $e_f(i, M) = e_{f'}(i, M)$, $i = 0, 1$. Suppose $t^{'}$ is the number of edges in $\{x_1y_1, x_2y_2, \ldots, x_{2n}y_{2n}\}$ with endpoints labelled differently under $f^{'}$. If e_0 is not incident with any of x_1, y_1 and w, then $t^{'} = 3$; if e_0 is incident with w, then $t^{'} = 2$; if e_0 is incident with x_1 or y_1, then $t^{'} = 1$. In any case above, by Case 1, we can conclude that G is E_2-cordial.

Theorem 6. Suppose G is a graph of order $4n+3$. If $V(G) = \{x_1, \ldots, x_{2n+1}, y_1, \ldots, y_{2n+1}, w\}$ and $\{x_1y_1, x_2y_2, \ldots, x_{2n+1}y_{2n+1}\} \subseteq E(G)$, then G is E_2-cordial.

Proof. Let N be a subgraph of G such that $E(N) = E(G)/\{x_1y_1, x_2y_2, \ldots, x_{2n+1}y_{2n+1}\}$ and $V(N) = V(G)$. We first define the edge label of N by $f : E(N) \to \{0, 1\}$ and $|e_f(1, N) - e_f(0, N)| \leq 1$, then we induce the vertex label of H by $f(v) \equiv \sum_{x \in N(v)} f(vx) (\text{mod } 2)$ and $f(v) \in \{0, 1\}$. Suppose E_{11} and E_{00} are edge sets defined in Theorem 4 and $E_{10} = \{x_{k+m+1}y_{k+m+1}, \ldots, x_{2n+1}y_{2n+1}\}$ satisfying $f(x_i) \neq f(y_i), i = k + m + 1, \ldots, 2n + 1$. In order to get an E_2-cordial label of G, we shall label the edges in E_{11}, E_{00} and E_{10}. For this, we distinguish two cases.

Case 1. $t = 2n + 1 - k - m > 0$.

Subcase 1. t is even. It follows that $f(w) = 0$ and k and m are of different parity. If m is odd, $e_f(0, N) = e_f(1, N)$ or m is odd, $e_f(0, N) = e_f(1, N) - 1$, $\frac{k}{2}$ edges in E_{11}, $\frac{m-1}{2}$ in E_{00} and $\frac{t}{2} + 1$ in E_{10} are labelled with 0 and others with 1. If m is odd and $e_f(0, N) = e_f(1, N) + 1$, $\frac{k}{2}$ edges in E_{11}, $\frac{m-1}{2}$ in E_{00} and $\frac{t}{2}$ in E_{10} are labelled with 0 and others with 1. Conversely, if k is odd, we can find an E_2-cordial label of G in a similar way.

Subcase 2.t is odd. It is obvious that $f(w) = 1$ and k and m are of the same parity. If k and m are both even and $e_f(0, N) \leq e_f(1, N)(e_f(0, N) >$

$e_f(1,N)$), Then $\frac{k}{2}$ edges in E_{11}, $\frac{m}{2}$ in E_{00} and $\frac{t+1}{2}(\frac{t-1}{2})$ in E_{10} are labelled with 0 and others with 1. If k and m are both odd and $e_f(0,N) \leq e_f(1,N)$, then $\frac{k+1}{2}$ edges in E_{11}, $\frac{m+1}{2}$ in E_{00} and $\frac{t-1}{2}$ in E_{10} are labelled with 0 and others with 1. Similarly, If k and m are both odd and $e_f(0,N) > e_f(1,N)$, then $\frac{k-1}{2}$ edges in E_{11}, $\frac{m-1}{2}$ in E_{00} and $\frac{t+1}{2}$ in E_{10} are labelled with 0 and others with 1. In any case, the result is an E_2-cordial label of G.

Case 2. $t = 0$. It follows that $f(w) = 0$ and k and m are of different parity. Suppose k is odd.

Subcase 1. $e_f(0,N) = e_f(1,N)$ or $e_f(0,N) = e_f(1,N) - 1$. $\frac{k+1}{2}$ edges in E_{11} and $\frac{m}{2}$ in E_{00} are labelled with 0 and others with 1.

Subcase 2. $e_f(0,N) = e_f(1,N) + 1$. There exists an edge $e_0 \in E(N)$ labelled with 0. By changing the label of the edge e_0 from 0 to 1, we get another label f' of N under which $e_{f'}(0,N) = e_{f'}(1,N) - 1$. Denote by t' the number of edges in $\{x_1y_1, x_2y_2, \ldots, x_{2n+1}y_{2n+1}\}$ with endpoints labelled differently under f'. If e_0 is incident with w, then $t' = 1$, if not, then $t' = 2$. In either case, by Case 1, we can conclude that G is E_2-cordial.

If m is odd, we can prove that G is E_2-cordial in a similar way.

Remark. In this section, we prove the E_2-cordiality of lots of graphs. For example, by Theorem 4, Theorem 5 and Theorem 6, the Cartesian product $P_m \times P_n$ is E_2-cordial if and only if $mn \neq 4n + 2$.

References

[1] I. Cahit and R. Yilmaz, E_3-cordial graphs, Ars Combin., 54(2000), 119-127.

[2] M.V. Bapat and N.B. Limaye, Edge-three cordial graphs arising from complete graphs, J. Combin, Math. Combin. comput, 56(2006), 147-169.

[3] N. Sridharan, K. Uma Rani, E_k-cordial labelings of graphs, in: R. Balakrishnan, R.J. Wilson, C. Sethuraman(Eds.), In Graph theory and its Applications, Narosa Publishing house, New Delhi, 2004, PP.96-102.