On the signless Laplacian spectra of bicyclic and tricyclic graphs *

Muhuo Liu[1,2], Bolian Liu[2]

[1] Department of Applied Mathematics, South China Agricultural University, Guangzhou, P. R. China, 510642

[2] School of Mathematic Science, South China Normal University, Guangzhou, P. R. China, 510631

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Abstract: In this paper, the first two (resp. four) largest signless Laplacian spectral radii together with the corresponding graphs in the class of bicyclic (resp. tricyclic) graphs of order n are determined, and the first two (resp. four) largest signless Laplacian spreads together with the corresponding graphs in the class of bicyclic (resp. tricyclic) graphs of order n are identified.

Key words: Signless Laplacian spectral radii, signless Laplacian spread, bicyclic graph, tricyclic graph.

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1 Introduction

Throughout the paper, $G = (V, E)$ is a connected undirected simple graph with $|V| = n$ and $|E| = m$. If $m = n + c - 1$, then G is called a *c-cyclic graph*. Especially, if $c = 1, 2, 3$, then G is called a unicyclic, bicyclic, or tricyclic graph, respectively. The neighbors of a vertex v is denoted by $N(v)$. Write $d(v)$ for the degree of vertex v. Specially, $\Delta = \Delta(G)$ denotes the maximum degree of G.

Let $A(G)$ denote the *adjacency matrix* of G. Let $D(G)$ be the diagonal matrix whose (i, i)-entry is $d(v_i)$. The *Laplacian matrix* of G is $L(G) = D(G) - A(G)$, and the *signless Laplacian matrix* of G is $Q(G) = D(G) +$

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$A(G)$. The largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the *adjacency, Laplacian, and signless Laplacian spectral radii*, respectively. It is well-known that graph spectrum has great important applications in many fields. Several graph spectra, i.e., spectra of $A(G)$, $L(G)$ and $Q(G)$, had been defined in [1]. The spectra of $A(G)$, $L(G)$ are well studied (for instance see [2-10]), but the spectrum of $Q(G)$ seems to be less well known. It is not until recent years, some researchers found that the spectrum of $Q(G)$ has a strong connection with the structures of graph (see [11, 12]). For the new results on the signless Laplacian spectrum, one can refer to [11-14].

The largest adjacency spectral radius in the class of unicyclic graphs of given order was firstly determined in [2]. Following this, Guo [3] determined the first six largest adjacency spectral radii in the class of unicyclic graphs of given order. After then, the first three largest adjacency spectral radii in the class of bicyclic graphs of given order were given in [4]. Similarly, the researchers had investigated the Laplacian spectral radius in the class of unicyclic, bicyclic and tricyclic graphs. Up to now, the first to thirteenth largest Laplacian spectral radii were given in the class of unicyclic graphs of given order in [5–7]. After then, the first eight largest Laplacian spectral radii in the class of bicyclic graphs of given order were given in [8, 9]. Recently, the first nineteen largest Laplacian spectral radii in the class of tricyclic graphs of given order were determined in [10]. Motivated by the above results on $A(G)$ and $L(G)$, we shift our goals to the investigation of signless Laplacian spectral radius. Recently, we had identified the first four largest signless Laplacian spectral radii in the class of unicyclic graphs of given order in [15]. This paper will present the first two (resp. four) largest signless Laplacian spectral radii together with the corresponding graphs in the class of bicyclic (resp. tricyclic) graphs of given order.

Since $A(G)$ is symmetric, the eigenvalues of $A(G)$ can be arranged as follows: $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_n(G)$. The *adjacency spread* of the graph G, denoted by $SA(G)$, is defined as (see [16]):

$$SA(G) = \rho_1(G) - \rho_n(G).$$

It is well known that $L(G)$ is positive semidefinite so that its eigenvalues can be arranged as follows: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{n-1}(G) \geq \lambda_n(G) = 0$, where $\lambda_{n-1}(G) > 0$ if and only if G is connected and is called the *algebraic connectivity* of the graph G. Because $\lambda_n(G) = 0$, the *Laplacian spread* of the graph G, denoted by $SL(G)$, is defined as [17]

$$SL(G) = \lambda_1(G) - \lambda_{n-1}(G).$$

Note that $Q(G)$ is also positive semidefinite [14] and its eigenvalues can be arranged as: $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq 0$. Thus, the *signless*

Laplacian spread of the graph G, denoted by $SQ(G)$, is defined as

$$SQ(G) = \mu_1(G) - \mu_n(G).$$

This definition firstly appeared in the conjecture 25 of the literature [12], but well studied in [13] by the name of Q-spread. In [14], it is called signless Laplacian spread to distinguish the known notations adjacency spread and Laplacian spread.

The adjacency spread of a graph had received much attention. In [18], Petrović determined all connected graphs with adjacency spread at most 4. In [16, 19], some lower and upper bounds for the adjacency spread of a graph were given. After then, the maximum adjacency spreads among all unicyclic graphs and all bicyclic graphs of given order were determined in [20] and [21], respectively. Similarly, the maximum and minimum Laplacian spreads among all trees of given order were identified in [17], the maximum and minimum Laplacian spreads among all unicyclic graphs of given order were determined in [22] and [23], respectively, the maximum Laplacian spread among tricyclic graphs of given order was identified in [24]. However, the signless Laplacian spread seems less well-known because it was introduced somewhat later [13, 14]. Up to now, there are only very limited results on the signless Laplacian spread. Some lower and upper bounds for the signless Laplacian spread of a graph were given in [13, 14], and the largest signless Laplacian spread in the class of unicyclic graphs of given order was determined in [14]. In this paper, by using different method from [13, 14], we determine the first two (resp. four) largest signless Laplacian spreads together with the corresponding graphs in the class of bicyclic (resp. tricyclic) graphs of given order.

This paper is organized as follows: The first two (resp. four) largest signless Laplacian spectral radii among the class of bicyclic (resp. tricyclic) graphs of given order are presented in Section 2. In Section 3, we determine the first two (resp. four) largest signless Laplacian spreads among the class of bicyclic (resp. tricyclic) graphs of given order.

2 The first two (resp. four) largest signless Laplacian spectral radii of bicyclic (resp. tricyclic) graphs

In the sequel, the notations $\mathcal{B}_n$, $\mathcal{T}_n$ are used to denote the class of bicyclic graphs and tricyclic graphs of order n, respectively.

Lemma 2.1 [25] *$\mu_1(G) \leq max\{d(v) + m(v) : v \in V\}$, where $m(v) = \sum\limits_{u \in N(v)} d(u)/d(v)$.*

Lemma 2.2 [26,27] *If G is a graph with at least one edge, then $\mu_1(G) \geq \lambda_1(G) \geq \Delta + 1$. If G is connected, the first equality holds if and only if G is bipartite.*

Proposition 2.1 *Suppose $c \geq 0$ and G is a c-cyclic graph on n vertices with $\Delta \leq n-2$. If $n \geq max\{c+5, 2c+3\}$, then $\mu_1(G) \leq n$.*

Proof. By Lemma 2.1, we only need to prove that $max\{d(v)+m(v) : v \in V\} \leq n$.

Suppose $max\{d(v)+m(v) : v \in V\}$ occurs at the vertex u. Three cases arise $d(u)=1$, $d(u)=2$, or $3 \leq d(u) \leq n-2$.

Case 1. $d(u)=1$. Suppose $v \in N(u)$. Since $m(u) = d(v) \leq \Delta \leq n-2$, thus $d(u)+m(u) \leq n-1 < n$.

Case 2. $d(u)=2$. Suppose that $v,w \in N(u)$. Note that G is a c-cyclic graph, then $|N(v) \cap N(w)| \leq c+1$ and $|N(v) \cup N(w)| \leq n$. Therefore, $d(u)+m(u) = 2 + \frac{d(v)+d(w)}{2} \leq 2 + \frac{n+c+1}{2} \leq n$.

Case 3. $3 \leq d(u) \leq n-2$. Suppose G has m edges. Note that $3 \leq d(u) \leq n-2$, then $d(u)+m(u) \leq d(u) + \frac{2m-d(u)-1}{d(u)} = d(u)-1+\frac{2m-1}{d(u)}$. Next we shall prove that $d(u)-1+\frac{2m-1}{d(u)} \leq n$, equivalently, $d(u)(n+1-d(u)) \geq 2m-1$. Once this is proved, we are done. Let $f(x) = (n+1-x)x$.

When $x \in [3, \frac{n+1}{2}]$, since $f'(x) = n+1-2x \geq 0$, then $f(x) \geq f(3) = 3(n-2) \geq 2(n+c-1)-1 = 2m-1$.

When $x \in [\frac{n+1}{2}, n-2]$, since $f'(x) = n+1-2x \leq 0$, then $f(x) \geq f(n-2) = 3(n-2) \geq 2(n+c-1)-1 = 2m-1$.

By combining the above arguments, the assertion follows. ∎

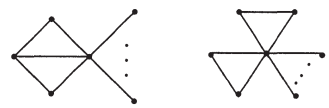
Corollary 2.1 (1) *Suppose $G \in \mathcal{B}_n$. If $n \geq 7$ and $\Delta \leq n-2$, then $\mu_1(G) \leq n$.* (2) *Suppose $G \in \mathcal{T}_n$. If $n \geq 9$ and $\Delta \leq n-2$, then $\mu_1(G) \leq n$.*

It is well-known that $\lambda_1(G) \leq n$ (for example see [26]). For $\mu_1(G)$, we have

Corollary 2.2 *Suppose $c \geq 0$ and G is a c-cyclic graph with order $n > max\{c+5, 2c+3\}$, then* (1) *$\mu_1(G) > n$ if and only if $\Delta = n-1$ and G contains cycle of length odd.* (2) *$\mu_1(G) = n$ if and only if $\Delta = n-1$ and G is bipartite.*

Proof. It can be proved similarly with Proposition 2.1 that if $\Delta \leq n-2$, then $\mu_1(G) < n$ because $n > max\{c+5, 2c+3\}$. Next we only need to consider the case of $\Delta = n-1$. If $\Delta = n-1$ and G is bipartite, then $\mu_1(G) = n$ follows from Lemma 2.2. If $\Delta = n-1$ and G contains cycle of length odd, by Lemma 2.2 $n = \lambda_1(G) < \mu_1(G)$. ∎

In the following, $V \backslash \{u\}$ is written as $V-u$ for convenience. Let H_1, H_2 be the bicyclic graphs on $n \geq 7$ vertices as shown in Fig. 1.



H_1 H_2

Fig. 1

Theorem 2.1 *Suppose $G \in \mathcal{B}_n$ and $n \geq 7$. Then, (1) $\mu_1(G) \leq \mu_1(H_1)$, the equality holds if and only if $G \cong H_1$, where $\mu_1(H_1)$ is the maximum root of the equation $\lambda^3 - (n+4)\lambda^2 + 4n\lambda - 8 = 0$. (2) If $G \ncong H_1$, then $\mu_1(G) \leq \mu_1(H_2)$, the equality holds if and only if $G \cong H_2$, where $\mu_1(H_2)$ is the maximum root of the equation $\lambda^3 - (n+3)\lambda^2 + 3n\lambda - 8 = 0$.*

Proof. Let $\Phi(G, \lambda)$=det$(\lambda I - Q(G))$ denote the signless Laplacian characteristic polynomial of G. By a straightforward computation, we have

$$\Phi(H_1, \lambda) = (\lambda - 1)^{n-4}(\lambda - 2)(\lambda^3 - (n+4)\lambda^2 + 4n\lambda - 8). \tag{1}$$

$$\Phi(H_2, \lambda) = (\lambda - 1)^{n-4}(\lambda - 3)(\lambda^3 - (n+3)\lambda^2 + 3n\lambda - 8). \tag{2}$$

Since $\Phi(H_1, n) = (n-1)^{n-4}(16-8n) < 0$ and $\lim\limits_{\lambda \longrightarrow +\infty} \Phi(H_1, \lambda) = +\infty$, thus $\mu_1(H_1) > n$. With the same reason, $\mu_1(H_2) > n$. Suppose that $G \in \mathcal{B}_n$ and $\mu_1(G) > n$, Corollary 2.1 implies that $\Delta = n - 1$. Next we shall show that $G \cong H_1$ or $G \cong H_2$.

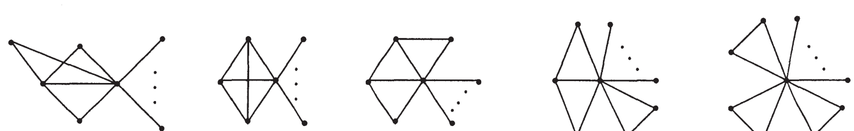
Assume that $u \in V(G)$ such that $d(u) = n - 1$. It is easy to see that $uv \in E(G)$ holds for every $v \in V - u$. Since $G \in \mathcal{B}_n$, then $1 \leq max\{|N(u) \cap N(v)| : v \in V - u\} \leq 2$. Thus, two cases should be considered.

Case 1. $max\{|N(u) \cap N(v)| : v \in V - u\} = 2$. It is easy to see that $G \cong H_1$.

Case 2. $max\{|N(u) \cap N(v)| : v \in V - u\} = 1$. It is easy to see that $G \cong H_2$.

When $\lambda \geq n \geq 7$, since $\Phi(H_2, \lambda) - \Phi(H_1, \lambda) = (\lambda - 1)^{n-4}((\lambda - n)\lambda + 8) > 0$, thus $\mu_1(H_1) > \mu_1(H_2)$ because $\mu_1(H_1)$, $\mu_1(H_2) > n$.

By Eqs. (1)-(2) and the above arguments, the conclusion follows. ∎



F_1 F_2 F_3 F_4 F_5

Fig. 2

Let F_1, F_2, F_3, F_4, F_5 be the tricyclic graphs on $n \geq 9$ vertices as shown in Fig. 2.

Lemma 2.3 *Suppose $G \in \mathcal{T}_n$ and $n \geq 9$. If $\mu_1(G) > n$, then G should be the graph F_1, F_2, F_3, F_4 or F_5.*

Proof. Since $G \in \mathcal{T}_n$ and $\mu_1(G) > n$, Corollary 2.1 implies that $\Delta = n-1$. Next we shall show that $G \cong F_i$, where $1 \leq i \leq 5$.

Assume that $u \in V(G)$ such that $d(u) = n-1$. It is easy to see that $uv \in E(G)$ holds for every $v \in V-u$. Since $G \in \mathcal{T}_n$, then $1 \leq max\{|N(u) \cap N(v)| : v \in V-u\} \leq 3$. Thus, three cases should be considered.

Case 1. $max\{|N(u) \cap N(v)| : v \in V-u\} = 3$. It is easy to see that $G \cong F_1$.

Case 2. $max\{|N(u) \cap N(v)| : v \in V-u\} = 2$. It is easy to see that $G \cong F_2$ or F_3 or F_4.

Case 3. $max\{|N(u) \cap N(v)| : v \in V-u\} = 1$. It is easy to see that $G \cong F_5$. ∎

Lemma 2.4 *If $n \geq 9$, then $\mu_1(F_1) = \mu_1(F_2) > \mu_1(F_3) > \mu_1(F_4) > \mu_1(F_5) > n$.*

Proof. By an elementary calculation, we have

$$\Phi(F_1,\lambda) = (\lambda-1)^{n-5}(\lambda-2)^2(\lambda^3-(n+5)\lambda^2+5n\lambda-12). \tag{3}$$

$$\Phi(F_2,\lambda) = (\lambda-1)^{n-5}(\lambda-2)^2(\lambda^3-(n+5)\lambda^2+5n\lambda-12). \tag{4}$$

$$\Phi(F_3,\lambda) = (\lambda-1)^{n-5}(\lambda-3)(\lambda^4-(n+6)\lambda^3+(6n+7)\lambda^2$$
$$-(7n+12)\lambda+20). \tag{5}$$

$$\Phi(F_4,\lambda) = (\lambda-1)^{n-5}(\lambda-2)(\lambda^4-(n+7)\lambda^3+(7n+12)\lambda^2$$
$$-(12n+12)\lambda+40). \tag{6}$$

$$\Phi(F_5,\lambda) = (\lambda-1)^{n-5}(\lambda-3)^2(\lambda^3-(n+3)\lambda^2+3n\lambda-12). \tag{7}$$

By Eqs. (3) and (4), $\mu_1(F_1) = \mu_1(F_2)$. Since $n \geq 9$ and $\Phi(F_1,n) = -12(n-1)^{n-5}(n-2)^2 < 0$ and $\lim_{\lambda \longrightarrow +\infty} \Phi(F_1,\lambda) = +\infty$, thus $\mu_1(F_1) > n$. With the same reason, $\mu_1(F_i) > n$ also holds for $2 \leq i \leq 5$.

We divide the proof into the next three processes.

(1) $\mu_1(F_2) > \mu_1(F_3)$. Rewrite Eq. (4) as $\Phi(F_2,\lambda) = (\lambda-1)^{n-5}f_1(\lambda)$, where $f_1(\lambda) = \lambda^5-(n+9)\lambda^4+(9n+24)\lambda^3-(24n+32)\lambda^2+(20n+48)\lambda-48$. Thus, $\mu_1(F_2)$ equals the maximum root of the equation $f_1(\lambda) = 0$. Rewrite Eq. (5) as $\Phi(F_3,\lambda) = (\lambda-1)^{n-5}f_2(\lambda)$, where $f_2(\lambda) = \lambda^5-(n+9)\lambda^4+$

$(9n+25)\lambda^3-(25n+33)\lambda^2+(21n+56)\lambda-60$. Thus, $\mu_1(F_3)$ equals the maximum root of the equation $f_2(\lambda)=0$. Let $\psi_1(\lambda)=f_2(\lambda)-f_1(\lambda)=\lambda^3-(n+1)\lambda^2+(n+8)\lambda-12$. When $\lambda\geq n\geq 9$, since $\psi_1'(\lambda)=3\lambda^2-2(n+1)\lambda+(n+8)=\lambda(3\lambda-2n-2)+n+8>0$, then $\psi_1(\lambda)\geq\psi_1(n)=8n-12>0$. Thus, when $\lambda\geq n\geq 9$, $f_2(\lambda)>f_1(\lambda)$. This implies that $\mu_1(F_2)>\mu_1(F_3)$ because $\mu_1(F_2)$, $\mu_1(F_3)>n$.

(2) $\mu_1(F_3)>\mu_1(F_4)$. Rewrite Eq. (6) as $\Phi(F_4,\lambda)=(\lambda-1)^{n-5}f_3(\lambda)$, where $f_3(\lambda)=\lambda^5-(n+9)\lambda^4+(9n+26)\lambda^3-(26n+36)\lambda^2+(24n+64)\lambda-80$. Thus, $\mu_1(F_4)$ equals the maximum root of the equation $f_3(\lambda)=0$. Let $\psi_2(\lambda)=f_3(\lambda)-f_2(\lambda)=\lambda^3-(n+3)\lambda^2+(3n+8)\lambda-20$. When $\lambda\geq n\geq 9$, since $\psi_2'(\lambda)=3\lambda^2-2(n+3)\lambda+(3n+8)=\lambda(3\lambda-2n-6)+3n+8>0$, thus $\psi_2(\lambda)\geq\psi_2(n)=8n-20>0$. Thus, when $\lambda\geq n\geq 9$, $f_3(\lambda)>f_2(\lambda)$. This implies that $\mu_1(F_3)>\mu_1(F_4)$ because $\mu_1(F_3)$, $\mu_1(F_4)>n$.

(3) $\mu_1(F_4)>\mu_1(F_5)$. Rewrite Eq. (7) as $\Phi(F_5,\lambda)=(\lambda-1)^{n-5}f_4(\lambda)$, where $f_4(\lambda)=\lambda^5-(n+9)\lambda^4+(9n+27)\lambda^3-(27n+39)\lambda^2+(27n+72)\lambda-108$. Thus, $\mu_1(F_5)$ equals the maximum root of the equation $f_4(\lambda)=0$. Let $\psi_3(\lambda)=f_4(\lambda)-f_3(\lambda)=\lambda^3-(n+3)\lambda^2+(3n+8)\lambda-28$. When $\lambda\geq n\geq 9$, since $\psi_3'(\lambda)=3\lambda^2-2(n+3)\lambda+(3n+8)=\lambda(3\lambda-2n-6)+3n+8>0$, thus $\psi_3(\lambda)\geq\psi_3(n)=8n-28>0$. Thus, when $\lambda\geq n\geq 9$, $f_4(\lambda)>f_3(\lambda)$. This implies that $\mu_1(F_4)>\mu_1(F_5)$ because $\mu_1(F_4)$, $\mu_1(F_5)>n$. ∎

By combining Corollary 2.1, Lemmas 2.3-2.4, and Eqs. (3)-(7), we have

Theorem 2.2 *Suppose $G\in\mathcal{T}_n$ and $n\geq 9$. Then, (1) $\mu_1(G)\leq\mu_1(F_1)$, the equality holds if and only if $G\cong F_1$ or $G\cong F_2$, where $\mu_1(F_1)$ is the maximum root of the equation $\lambda^3-(n+5)\lambda^2+5n\lambda-12=0$. (2) If $G\notin\{F_1,F_2\}$, then $\mu_1(G)\leq\mu_1(F_3)$, the equality holds if and only if $G\cong F_3$, where $\mu_1(F_3)$ is the maximum root of the equation $\lambda^4-(n+6)\lambda^3+(6n+7)\lambda^2-(7n+12)\lambda+20=0$. (3) If $G\notin\{F_1,F_2,F_3\}$, then $\mu_1(G)\leq\mu_1(F_4)$, the equality holds if and only if $G\cong F_4$, where $\mu_1(F_4)$ is the maximum root of the equation $\lambda^4-(n+7)\lambda^3+(7n+12)\lambda^2-(12n+12)\lambda+40=0$. (4) If $G\notin\{F_1,F_2,F_3,F_4\}$, then $\mu_1(G)\leq\mu_1(F_5)$, the equality holds if and only if $G\cong F_5$, where $\mu_1(F_5)$ is the maximum root of the equation $\lambda^3-(n+3)\lambda^2+3n\lambda-12=0$.*

3 The first two (resp. four) largest signless Laplacian spreads of bicyclic (resp. tricyclic) graphs

Lemma 3.1 *If G is a bicyclic graph on $n\geq 12$ vertices with $\Delta\leq n-2$, then $\mu_1(G)\leq n-0.5$.*

Proof. By Lemma 2.1, we only need to prove that $max\{d(v) + m(v) : v \in V\} \leq n - 0.5$.

Suppose $max\{d(v) + m(v) : v \in V\}$ occurs at the vertex u. Three cases arise $d(u) = 1$, $d(u) = 2$, or $3 \leq d(u) \leq n - 2$.

Case 1. $d(u) = 1$. Suppose $v \in N(u)$. Since $m(u) = d(v) \leq \Delta \leq n - 2$, thus $d(u) + m(u) \leq n - 1 < n - 0.5$.

Case 2. $d(u) = 2$. Suppose that $v, w \in N(u)$. Note that G is a bicyclic graph, then $|N(v) \cap N(w)| \leq 3$ and $|N(v) \cup N(w)| \leq n$. Therefore, $d(u) + m(u) = 2 + \frac{d(v)+d(w)}{2} \leq 2 + \frac{n+3}{2} \leq n - 0.5$.

Case 3. $3 \leq d(u) \leq n - 2$. Note that $3 \leq d(u) \leq n - 2$ and G has $n + 1$ edges, then $d(u) + m(u) \leq d(u) + \frac{2n+1-d(u)}{d(u)} = d(u) - 1 + \frac{2n+1}{d(u)}$. Next we shall prove that $d(u) - 1 + \frac{2n+1}{d(u)} \leq n - 0.5$, equivalently, $d(u)(n + 0.5 - d(u)) \geq 2n + 1$. Let $f(x) = (n + 0.5 - x)x$.

When $x \in [3, \frac{2n+1}{4}]$, since $f'(x) = n + 0.5 - 2x \geq 0$, then $f(x) \geq f(3) = 3(n - 2.5) \geq 2n + 1$.

When $x \in [\frac{2n+1}{4}, n - 2]$, since $f'(x) = n + 0.5 - 2x \leq 0$, then $f(x) \geq f(n - 2) = 2.5(n - 2) \geq 2n + 1$.

This completes the proof of this result. ∎

Theorem 3.1 *If G is a bicyclic graph on $n \geq 12$ vertices and $G \notin \{H_1, H_2\}$, then $SQ(H_1) > SQ(H_2) > SQ(G)$.*

Proof. Let $f_5(\lambda) = \lambda^3 - (n+4)\lambda^2 + 4n\lambda - 8$ and $f_6(\lambda) = \lambda^3 - (n+3)\lambda^2 + 3n\lambda - 8$. Since $f_5(0) = -8 < 0$, $f_5(1) = 3n - 11 > 0$, $f_5(n) = -8 < 0$ and $f_5(n + 1) = n^2 - 2n - 11 > 0$, by Eq. (1) we can conclude that $0 < \mu_n(H_1) < 1$ and $n < \mu_1(H_1) < n + 1$.

Similarly, since $f_6(0) = -8 < 0$, $f_6(0.5) = \frac{1}{8}(10n - 69) > 0$, $f_6(n) = -8 < 0$ and $f_6(n + 1) = n^2 - n - 10 > 0$, by Eq. (2) it follows that $0 < \mu_n(H_2) < 0.5$ and $n < \mu_1(H_2) < n + 1$, this implies that $SQ(H_2) = \mu_1(H_2) - \mu_n(H_2) > n - 0.5$.

When $0 < \lambda < 1$, $f_5(\lambda) - f_6(\lambda) = -(\lambda - n)\lambda > 0$, thus $\mu_n(H_1) < \mu_n(H_2)$ because $0 < \mu_n(H_1) < 1$ and $0 < \mu_n(H_2) < 0.5$. By Theorem 2.1, we have

$$SQ(H_1) = \mu_1(H_1) - \mu_n(H_1) > \mu_1(H_2) - \mu_n(H_2) = SQ(H_2) > n - 0.5.$$

Moreover, note that $\mu_n(G) \geq 0$ and $G \notin \{H_1, H_2\}$, by Lemma 3.1 it follows that $SQ(G) = \mu_1(G) - \mu_n(G) \leq \mu_1(G) \leq n - 0.5 < SQ(H_2)$.

By combining the above arguments, the result follows. ∎

Remark 1. By Theorems 2.1 and 3.1, the graphs which share the first two largest signless Laplacian spectral radii even share the first two largest signless Laplacian spreads in the class of bicyclic graphs on $n \geq 12$ vertices.

It can be proved similarly with Lemma 3.1 that

Lemma 3.2 *If G is a tricyclic graph on $n \geq 16$ vertices with $\Delta \leq n-2$, then $\mu_1(G) \leq n - 0.5$.*

Lemma 3.3 *If $n \geq 16$, then $SQ(F_1) = SQ(F_2) > SQ(F_3) > SQ(F_4) > SQ(F_5) > n - 0.5$.*

Proof. By Eqs. (3) and (4), it follows that $SQ(F_1) = SQ(F_2)$. We divide the proof into the next three processes.

(1) $SQ(F_2) > SQ(F_3)$. If $16 \leq n \leq 30$, it can be checked readily that $SQ(F_2) > SQ(F_3)$ by Eqs. (4) and (5). Next we may suppose that $n \geq 31$. Note that $f_1(0) = -48 < 0$, $f_1(1) = 4n - 16 > 0$, $f_1(3) = 6n - 30 > 0$, $f_1(5) = -108 < 0$ and $f_1(n+5) = 5n^4+55n^3+183n^2+153n-108 > 0$, then $0 < \mu_n(F_2) < 1$, $\mu_{n-1}(F_2) = 1$ by Eq. (4). Note that $f_2(0) = -60 < 0$, $f_2(1) = 4n - 20 > 0$, $f_2(2) = 8 - 2n < 0$ and $f_2(4) = 4n - 44 > 0$, $f_2(8) = 6980 - 920n < 0$ and $f_2(n+5) = 5n^4+55n^3+188n^2+206n+20 > 0$, then $0 < \mu_n(F_3) < 1$, $\mu_{n-1}(F_3) = 1$ by Eq. (5).

Let α_1 be the minimum root of $\psi_1(\lambda) = 0$. Since $\psi_1(0) = -12 < 0$, $\psi_1(0.625) = \frac{1}{512}(120n - 3659) > 0$, $\psi_1(1) = -4 < 0$ and $\psi_1(n) = 8n - 12 > 0$, then $0 < \alpha_1 < 0.625$.

It is easy to see that

$$f_1(\lambda) = (\lambda^2 - 8\lambda + 8)\psi_1(\lambda) + \gamma_1(\lambda), \tag{8}$$

$$f_2(\lambda) = (\lambda^2 - 8\lambda + 9)\psi_1(\lambda) + \gamma_1(\lambda), \tag{9}$$

where $\gamma_1(\lambda) = -4(2n - 13)\lambda^2 + 4(3n - 28)\lambda + 48$.

When $0 < \lambda < 0.625$, since $\gamma_1'(\lambda) = -8(2n - 13)\lambda + 4(3n - 28) > 4(3n - 28) - 0.625 \times 8(2n - 13) = 2n - 47 > 0$, then $\gamma_1(\lambda) > \gamma_1(0) = 48 > 0$. Recall that $0 < \alpha_1 < 0.625$, by Eqs. (8) and (9) it follows that $f_1(\alpha_1) = f_2(\alpha_1) = \gamma_1(\alpha_1) > 0$. This implies that $\mu_n(F_2), \mu_n(F_3) \in (0, \alpha_1)$. Moreover, when $0 < \lambda < \alpha_1$, since $f_2(\lambda) - f_1(\lambda) = \psi_1(\lambda) < 0$, then $\mu_n(F_2) < \mu_n(F_3)$. By Theorem 2.2, we have

$$SQ(F_2) = \mu_1(F_2) - \mu_n(F_2) > \mu_1(F_3) - \mu_n(F_3) = SQ(F_3).$$

(2) $SQ(F_3) > SQ(F_4)$. Note that $f_3(0) = -80 < 0$, $f_3(1) = 6n-34 > 0$, $f_3(2.5) = \frac{1}{32}(235 - 30n) < 0$, $f_3(3) = 4 > 0$, $f_3(5) = 90 - 30n < 0$ and $f_3(n + 5) = 5n^4 + 55n^3 + 193n^2 + 249n + 90 > 0$, then $0 < \mu_n(F_4) < 1$, $\mu_{n-1}(F_4) = 1$ by Eq. (6).

Let α_2 be the minimum root of $\psi_2(\lambda) = 0$. Since $\psi_2(\frac{6}{n}) = \frac{-2}{n^3}(n^3 - 6n^2 + 54n - 108) < 0$, $\psi_2(1) = 2n - 14 > 0$, $\psi_2(5) = 70 - 10n < 0$ and $\psi_2(n) = 8n - 20 > 0$, then $\frac{6}{n} < \alpha_2 < 1$.

It is easy to see that

$$f_2(\lambda) = (\lambda^2 - 6\lambda - 1)\psi_2(\lambda) + \gamma_2(\lambda), \tag{10}$$

$$f_3(\lambda) = (\lambda^2 - 6\lambda)\psi_2(\lambda) + \gamma_2(\lambda), \tag{11}$$

where $\gamma_2(\lambda) = -8(n-4)\lambda^2 + 8(3n-7)\lambda - 80$.

When $\frac{6}{n} < \lambda < 1$, since $\gamma_2'(\lambda) = -16(n-4)\lambda + 8(3n-7) > 8(3n-7) - 16(n-4) = 8n+8 > 0$, then $\gamma_2(\lambda) > \gamma_2(\frac{6}{n}) = \frac{16}{n^2}(4n^2-39n+72) > 0$. Recall that $\frac{6}{n} < \alpha_2 < 1$, by Eqs. (10) and (11) it follows that $f_2(\alpha_2) = f_3(\alpha_2) = \gamma_2(\alpha_2) > 0$. This implies that $\mu_n(F_3)$, $\mu_n(F_4) \in (0, \alpha_2)$. Moreover, when $0 < \lambda < \alpha_2$, since $f_3(\lambda) - f_2(\lambda) = \psi_2(\lambda) < 0$, then $\mu_n(F_3) < \mu_n(F_4)$. By Theorem 2.2, we have

$$SQ(F_3) = \mu_1(F_3) - \mu_n(F_3) > \mu_1(F_4) - \mu_n(F_4) = SQ(F_4).$$

(3) $SQ(F_4) > SQ(F_5)$. Note that $f_4(\lambda) = (\lambda-3)^2\varphi_1(\lambda)$, where $\varphi_1(\lambda) = \lambda^3 - (n+3)\lambda^2 + 3n\lambda - 12$. Since $\varphi_1(0) = -12 < 0$, $\varphi_1(0.5) = \frac{1}{8}(10n-101) > 0$, $\varphi_1(1) = 2n - 14 > 0$, $\varphi_1(3) = -12 < 0$ and $\varphi_1(n+1) = n^2 - n - 14 > 0$, then $0 < \mu_n(F_5) < 0.5$, $\mu_{n-1}(F_5) = 1$ by Eq. (7). By Lemma 2.4, it follows that $SQ(F_5) = \mu_1(F_5) - \mu_n(F_5) > n - 0.5$.

Let α_3 be the minimum root of $\psi_3(\lambda) = 0$. Since $\psi_3(\frac{8}{n}) = \frac{-4}{n^3}(n^3 + 48n - 128) < 0$, $\psi_3(1) = 2n - 22 > 0$, $\psi_3(5) = 62 - 10n < 0$ and $\psi_3(n) = 8n - 28 > 0$, then $\frac{8}{n} < \alpha_3 < 1$.

It is easy to see that

$$f_3(\lambda) = (\lambda^2 - 6\lambda)\psi_3(\lambda) + \gamma_3(\lambda), \tag{12}$$

$$f_4(\lambda) = (\lambda^2 - 6\lambda + 1)\psi_3(\lambda) + \gamma_3(\lambda), \tag{13}$$

where $\gamma_3(\lambda) = -8(n-5)\lambda^2 + 8(3n-13)\lambda - 80$.

When $\frac{8}{n} < \lambda < 1$, since $\gamma_3'(\lambda) = -16(n-5)\lambda + 8(3n-13) > 8(3n-13) - 16(n-5) = 8n - 24 > 0$, then $\gamma_3(\lambda) > \gamma_3(\frac{8}{n}) = \frac{16}{n^2}(7n^2 - 84n + 160) > 0$. Recall that $\frac{8}{n} < \alpha_3 < 1$, by Eqs. (12) and (13) it follows that $f_3(\alpha_3) = f_4(\alpha_3) = \gamma_3(\alpha_3) > 0$. This implies that $\mu_n(F_4)$, $\mu_n(F_5) \in (0, \alpha_3)$. Moreover, when $0 < \lambda < \alpha_3$, since $f_4(\lambda) - f_3(\lambda) = \psi_3(\lambda) < 0$, then $\mu_n(F_4) < \mu_n(F_5)$. By Theorem 2.2, we have

$$SQ(F_4) = \mu_1(F_4) - \mu_n(F_4) > \mu_1(F_5) - \mu_n(F_5) = SQ(F_5) > n - 0.5.$$

By combining the above arguments, the result follows. ∎

Note that $\mu_n(G) \geq 0$, by Lemmas 3.2-3.3 we can conclude that

Theorem 3.2 *If $G \in \mathcal{T}_n \setminus \{F_1, F_2, F_3, F_4, F_5\}$ and $n \geq 16$, then $SQ(F_1) = SQ(F_2) > SQ(F_3) > SQ(F_4) > SQ(F_5) > SQ(G)$.*

Remark 2. By Theorems 2.2 and 3.2, the graphs which share the first four largest signless Laplacian spectral radii even share the first four largest signless Laplacian spreads in the class of tricyclic graphs on $n \geq 16$ vertices.

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