Long Paths containing k-ordered vertices in Graphs[1]

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Abstract

Let G be a graph on n vertices. If for any ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$, that is, the vertices in S appear in order of the sequence $v_1, v_2, \ldots, v_k$, there exists a $v_1 - v_k$ (hamiltonian) path containing S in the given order, then G is k-ordered (hamiltonian) connected. In this paper we will show that if G is $(k+1)$-connected and k-ordered connected, then for any ordered set S, there exists a $v_1 - v_k$ path P containing S in the given order such that $|P| \geq \min\{n, \sigma_2(G) - 1\}$ where $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G); uv \notin E(G)\}$ when G is not complete, otherwise set $\sigma_2(G) = \infty$. Our result generalizes several related results known before.

Keywords: long paths, k-ordered sets, k-ordered connected, k-ordered hamiltonian connected

1. Introduction

All graphs considered in this paper are finite, simple and undirected. For any graph G, we use $V(G)$ or just V to denote its vertex set and $E(G)$ or just E to denote its edge set. Let $|G|$ or $|V|$ denote the cardinality of V. Let H and S be subgraphs of G or vertex subsets of G and $v \in V(G)$. We denote the set of vertices in S that are adjacent to some vertices in H by $N_S(H)$. Likewise, $N_S(v)$ will denote the set of vertices in S that are adjacent to v. Hence the degree of v with respect to S is $|N_S(v)|$ and is denoted by $d_S(v)$. We denote the degree of v with respect to G by $d_G(v)$ or $d(v)$. Let $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. For a graph G of order $n \geq 3$, we define $\sigma_2(G) = \min\{d_G(u) + d_G(v) : u, v \in V(G); uv \notin E(G)\}$ when G is not complete, otherwise set $\sigma_2(G) = \infty$. A cycle (path) containing all vertices of G is called a <u>hamiltonian cycle (path)</u>. A graph G is said to be <u>hamiltonian</u> if it possesses a hamiltonian cycle and to be <u>hamiltonian connected</u> if for any distinct vertices x, y in G there is a hamiltonian path connecting x and

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y.

Various degree conditions have been studied for the hamiltonicity and the hamiltonian connectivity of graphs. Two well-known results regarding the hamiltonian graphs are due to Dirac and Ore.

Theorem 1 (Dirac (1952) [1]). Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.

Theorem 2 (Ore (1960) [6]). Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian.

By increasing the lower bound for $\sigma_2(G)$ in Theorem 2 by just one, Ore proved the following:

Theorem 3 (Ore (1963) [7]). Let G be a graph on n vertices. If $\sigma_2(G) \geq n+1$, then G is hamiltonian connected.

Very recently, people became interested not only in finding long paths and cycles in graphs, but also in finding long paths and cycles containing any ordered set of k vertices. A vertex set $S = \{v_1, v_2, \ldots, v_k\}$ is said to be an ordered set if the vertices in S appear in the order of the sequence $v_1, v_2, \ldots, v_k$. A graph G is said to be k-ordered (hamiltonian) if for every ordered set of vertices S such that $|S| = k, (k \leq n)$, G contains a (hamiltonian) cycle C encountering S in the given order. A graph G is said to be k-ordered (hamiltonian) connected if for every ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ such that $|S| = k, (k \leq n)$, G contains a $v_1 - v_k$ (hamiltonian) path P encountering S in the given order. We define a requisite path as a $v_1 - v_k$ path containing S in the given order. Let $p(S)$ be the length of the longest requisite $v_1 - v_k$ path containing S. Set $p_o(G) = \min\{p(S) : \text{S is an ordered set with } |S| = k\}$. Of course, if a graph is hamiltonian (hamiltonian connected), then it is obviously k-ordered hamiltonian (k-ordered hamiltonian connected) when $2 \leq k \leq 3$. This observation led Ng and Schultz to investigate the degree conditions of k-ordered hamiltonian graphs and they obtained:

Theorem 4 (Ng and Schultz (1997) [5]). Let G be a graph of order $n \geq 3$ and let k be an integer with $3 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then G is k-ordered hamiltonian.

Faudree et al. improved Theorem 4 as follows:

Theorem 5 (Faudree et al. (2003) [2]). Let k be an integer with $3 \leq k \leq \frac{n}{2}$ and let G be a graph of order n. If $\sigma_2(G) \geq n + \frac{(3k-9)}{2}$, then G is k-ordered hamiltonian.

In the same paper, Ng and Schultz considered the k-ordered hamiltonian connectedness of graphs under the same conditions as in Theorem 4 but for $k \geq 4$ and got the following:

Theorem 6 (Ng and Schultz (1997) [5]). Let G be a graph of order $n \geq 4$ and let k be an integer with $4 \leq k \leq n$. If $\sigma_2(G) \geq n + 2k - 6$, then G is k-ordered hamiltonian connected.

Corollary 1 (Ng and Schultz (1997) [5]). Let G be a graph of order $n \geq 4$ and let k be an integer such that $4 \leq k \leq n$. If $\delta(G) \geq \frac{n}{2} + k - 3$, then G is k-ordered hamiltonian connected.

By assuming that n is sufficiently large, Faudree et al. reduced the lower bound for $\sigma_2(G)$ in Theorem 6 and proved the following result:

Theorem 7 (Faudree et al. (2003) [3]). Let G be a graph of sufficiently large order n and $k \geq 3$. If $\sigma_2(G) \geq n + \frac{3k-6}{2}$, then G is k-ordered hamiltonian connected.

The purpose of this paper is to investigate the minimum ordered path length for any $(k+1)$-connected and k-ordered connected graph of order n. Our result is as follows:

Theorem 8. Let G be a $(k+1)$-connected $(k \geq 2)$ graph on n vertices. If G is k-ordered connected, then $p_o(G) \geq \min\{n-1, \sigma_2(G) - 2\}$.

Notice that Theorem 8 gives a lower bound for the length of any longest path connecting two distinct vertices and containing S ($|S| = k$) in the given order for a k-ordered connected graph. Theorem 8 generalizes Theorem 3 if we take $k = 2$. Since a graph G is k-ordered connected if G is k-ordered hamiltonian, by Theorem 5 and the fact that $\sigma_2(G) \geq n + \frac{3k-6}{2}$ implies G is $(k+1)$-connected for $k \geq 3$, one can easily check that Theorem 8 also generalizes Theorem 7.

We will first construct the following graphs to demonstrate the sharpness of the bounds of Theorem 8 and then show several lemmas in Section 2. The proof of Theorem 8 will be given in Section 3.

That the lower bound for $p_o(G)$ in Theorem 8 is best possible is shown by the following example. The graph G is composed of 3 subgraphs. The first subgraph is a clique on k vertices less the edges of a cycle on k vertices $K_k - E(C_k)$. The next subgraph is a clique K_{k-1}. The last subgraph is a set of independent vertices, M. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the vertices of $K_k - E(C_k)$ and $E(C_k) = \{v_1v_2, v_2v_3, \cdots, v_{k-1}v_k, v_kv_1\}$. Let $V(K_{k-1}) = \{x_1, x_2, \ldots, x_{k-1}\}$ and $V(M) = \{u_1, u_2, \ldots, u_s\}$ where $s \geq k$. All possible edges exist between M and K_{k-1} and all possible edges exist between K_{k-1} and $K_k - E(C_k)$. The vertices of M are adjacent to only those vertices in $K_k - E(C_k)$ that possess an odd index. For $k \geq 7$, we have $\sigma_2(G) = d(u_i) + d(u_j) \leq 2((k-1) + \frac{k+1}{2}) = 3k - 1 = n - s + k$. Since $s \geq k$, $\sigma_2(G) \leq n$. Note that between every pair of consecutive vertices in S on P, there must be an element of K_{k-1}. Since P begins and ends in S and $|S| = k$, P must miss some $u_i \in M$ and $|P| \leq n - 1$. In this case,

$p_o(G) = 3k - 3 = \min\{n-1, \sigma_2(G) - 2\}$.

Also consider the following example which indicates that the connectivity condition $k+1$ in Theorem 8 is best possible. Let G be a graph composed of 3 cliques: K_t, K_s, and K_k, and $K_k - E(C_k)$ as in the previous example. Let $k-2 \leq s < 2k-7$ and $k-2 \leq t < 2k-7$. All possible edges exist between the vertices of K_t and K_k, K_s and K_k and K_k and $K_k - E(C_k)$. Let $V(K_k) = \{x_1, x_2, \ldots, x_k\}$ and $V(K_t) = \{t_1, t_2, \ldots, t_t\}$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the vertices of $K_k - E(C_k)$. It is easy to check that G is k-connected. Since $t \geq k-2$ and $s \geq k-2$, we can choose two nonadjacent vertices in $K_k - E(C_k)$ to be those in the minimum degree sum. Then $\sigma_2(G) = 2(k-3) + 2k = 4k - 6$. Then a longest requisite path P must miss all the vertices of either K_t or K_s. Thus $p_o(G) \leq 2k + max\{s,t\} - 1 < 4k - 8 = min\{n-1, \sigma_2(G) - 2\}$. Actually, G can be generalized by adding more copies of K_s to a graph of arbitrarily large order.

2. Several Lemmas

For a path P with a given orientation and $u \in V(P)$, let u^+ denote the first successor of u on P and u^- denote the first predecessor of u on P. Also, if $v \in V(G)$ then $N_P^+(v)$ and $N_P^-(v)$ denote the set of vertices succeeding the neighbors of v on P and the set of vertices preceding the neighbors of v on P respectively. For $u, v \in V(P)$, $P[u,v]$ denotes the subpath of the path P from u to v in the given direction. For $P[u^+, v]$ we write $P(u,v]$. Similarly, for $P[u, v^-]$, we write $P[u,v)$. We use $\overline{P}[u,v]$ to denote the subpath of P from u to v in the reverse order.

In this section, we assume that $p_o(G) < n-1$ and let P be a longest requisite path for any ordered set $S = \{v_1, v_2, \ldots, v_k\}$ with $|P| < n$, then there exists some component $H \subseteq G - P$. Let $N_P(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P and set $P_i = P(x_i, x_{i+1})$ where $1 \leq i \leq t-1$. We will first investigate some properties of P. For $u, v \in V(P)$, we call any $P(u,v)$ a <u>good segment</u> of P if $V(P(u,v)) \cap S = \emptyset$. Since G is $(k+1)$-connected, $t \geq k+1$. Since $|S| = k$ and the endpoints of P are by definition the first and last vertex in the ordered set S, by the pigeonhole principle, at least two of those segments of P defined by $N_P(H)$ are good. Let P_i for some $1 \leq i \leq t-1$, be a good segment. A vertex v in $V(P_i)$ is said to be <u>insertible</u> on P if v is adjacent to two consecutive vertices in $V(P - P_i)$. For any two vertices u, v in G and a subgraph H of G, we use uP_Hv to denote a longest path connecting u and v with all internal vertices in H. We will first present several lemmas regarding P.

Lemma 1. Let $u, v \in V(P)$. If $N_P^+(x) \cap N(x) \cap P[u,v] = \emptyset$, then $d_{P[u,v)}(x) \leq \frac{|P[u,v]|+1}{2}$.

Lemma 2. For every component $H \subseteq G - P$, $N_P^+(H) \cap N(H) = \emptyset$.

Proof: Suppose there exist $u, v \in V(H)$ such that without loss of generality $N_P^+(v) \cap N(u) \neq \emptyset$. Then by inserting uP_Hv, we get a requisite path that is longer than P; a contradiction. □

The following two lemmas give some structural properties for the vertices in a good segment.

Lemma 3. Let P_i $(1 \leq i \leq t)$ be a good segment and $x \in V(P_i)$.

(i) If for every $y \in V(P(x_i, x))$, y is insertible, then all vertices in $P(x_i, x)$ can be inserted into $V(P) - V(P_i)$.

(ii) If for every $y \in V(P(x, x_{i+1}))$, y is insertible, then all vertices in $P(x, x_{i+1})$ can be inserted into $V(P) - V(P_i)$.

Proof: We only prove (i) here and (ii) can be easily checked by a symmetric argument to that of the proof of (i).

The proof of (i) is by induction. If $|V(P(x_i, x))| = 1$, then the result holds by the definition of an insertible vertex. Suppose that $|V(P(x_i, x))| \geq 2$ and assume that the result holds for all integers p when $|V(P(x_i, x))| \leq p$. Now we consider $|V(P(x_i, x))| = p+1$. Since x^- is insertible, there are two consecutive vertices say w and w^+ in $V(P - P_i)$ such that $x^-w \in E(G)$ and $x^-w^+ \in E(G)$. When $N(y) \cap \{w, w^+\} = \emptyset$ for any $y \in V(P(x_i, x^-))$, as $|V(P(x_i, x^-))| = p$ and x^- can be inserted using w and w^+, the result holds by the induction hypothesis. When $N(y) \cap \{w, w^+\} \neq \emptyset$ for some $y \in V(P(x_i, x^-))$, then choose the first such vertex, say y_1, in $V(P(x_i, x^-))$ and we can insert all vertices in $P[y_1, x^-]$ into $V(P) - V(P_i)$ using w and w^+. Since $|V(P(x_i, y_1))| < p$, by the induction hypothesis and the choice of y_1, all vertices in $P(x_i, y_1)$ can be inserted into $V(P) - V(P_i)$. Hence (i) holds. □

Lemma 4. Let P_i be any good segment and $u_i \neq u_i'$ be two vertices in $V(P_i)$ with $u_i' \in V(P[u_i, x_{i+1}))$ such that all vertices in $V(P(x_i, u_i)) \cup V(P(u_i', x_{i+1}))$ are insertible.

(i) If there is a vertex x in $N(y) \cap V(P - P[x_i, x_{i+1}])$ for some y in $V(P(x_i, u_i])$, then $N(y') \cap \{x^+, x^-\} = \emptyset$ for any $y' \in V(P[u_i', x_{i+1}))$.

(ii) All vertices in $V(P(x_i, u_i)) \cup V(P(u_i', x_{i+1}))$ can be inserted into a path containing $P[v_1, x_i] \cup P[x_{i+1}, v_k]$.

Proof: By contradiction, suppose there are some $x \in V(P - P[x_i, x_{i+1}])$, $y \in V(P(x_i, u_i])$ and $y' \in V(P[u_i', x_{i+1}))$ such that $\{x^+, x^-\} \cap N(y') \neq \emptyset$ and $xy \in E(G)$. Choose such y and y' with $|V(P(x_i, y))| + |V(P(y', x_{i+1}))|$ as small as possible (that is, $N(x) \cap V(P(x_i, y)) = \emptyset$ and $(N(x^+) \cup N(x^-)) \cap V(P(y', x_{i+1})) = \emptyset$). If $x \in V(P[v_1, x_i))$, then we can find a requisite path $P' = P[v_1, x)xyP(y, y')y'x^+P(x^+, x_i)x_iP_Hx_{i+1}P(x_{i+1}, v_k]$ when $x^+y' \in$

$E(G)$ (see Figure 1) and
$P' = P[v_1, x^-)x^-y'\overline{P}(y', y)yxP(x, x_i)x_iP_Hx_{i+1}P(x_{i+1}, v_k]$ when $x^-y' \in E(G)$ (see Figure 2).

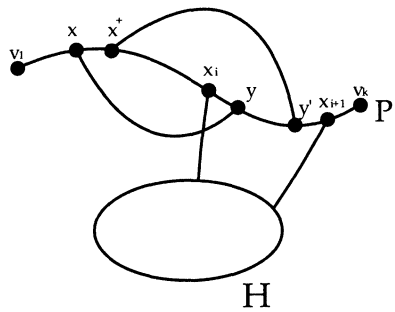

Figure 1

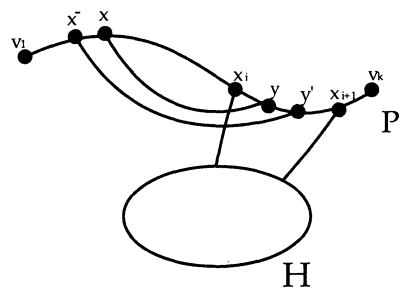

Figure 2

If $x \in V(P(x_{i+1}, v_k])$, then we can find a requisite path
$P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, x)xyP(y, y')y'x^+P(x^+, v_k]$, when $x^+y' \in E(G)$ (see Figure 3) and
$P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, x^-)x^-y'\overline{P}(y', y)yxP(x, v_k]$ when $x^-y' \in E(G)$ (see Figure 4).

In either case, by the choice of y, y' and Lemma 3, we can find a requisite path longer than P by inserting all vertices in $V(P(x_i, y)) \cup V(P(y', x_{i+1}))$ into P', contradicting the choice of P.

(ii) By Lemma 3 and Lemma 4(i), we can easily verify (ii). □

Next, we will use the parameter $D(G)$ as defined by Fraisse and Jung in [4]. For any 2-connected graph G, let $D(G)$ be the maximum integer m such that for any two distinct vertices u, v in G, there is a path of length at least m connecting u and v. For a complete graph K_n ($n \geq 2$), set $D(K_n) = n - 1$. If G has connectivity one, set $D(G) = \max\{D(G') :$

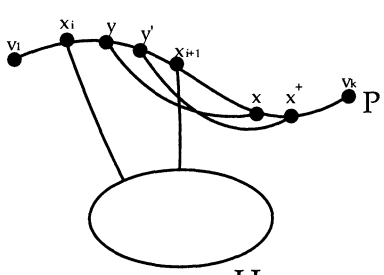

Figure 3

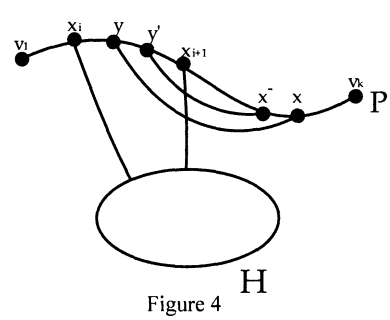

Figure 4

G' is an endblock of $G\}$. For an arbitrary graph, set $D(G) = \max\{D(G') : G'$ is a component of $G\}$.

Lemma 5 (**Fraisse and Jung (1989)** [4]). Let G be a noncomplete connected graph. Then there exist nonadjacent vertices v_1 and v_2 in G such that v_i is not a cut vertex of G and $D(G) \geq d(v_i) \quad (i = 1, 2)$.

Thus for a complete graph, $d(v) \leq D(G)$ for all $v \in V(G)$. If G is not 2-connected, let G_1 and G_2 be distinct endblocks of G. Then for any vertex $v_i \in V(G_i)$ that is not a cut vertex, there exists a path $P[v_i, v_j]$ such that $|P[v_i, v_j]| \geq D(G_1) + D(G_2) + 1$.

Let B_1 be an endblock of H with $D(B_1) = D(H)$. If H is 2-connected or $|H| \leq 2$, set $B_1 = H$. Let c_1 be the unique cut vertex of H in B_1 when H is not 2-connected and $|H| \geq 3$. Otherwise, let c_1 be an arbitrary vertex of H. Set $B = B_1 - \{c_1\}$.

Lemma 6. Let $P(u, v)$ be a good segment of P. Let $y_1 \neq y_2 \in V(H)$ with $\{y_1, y_2\} \cap V(B) \neq \emptyset$ and $uy_1 \in E$ and $vy_2 \in E$. Then there exists a good segment $P(u', v') \subseteq P(u, v)$ such that $V(P(u', v')) \cap N_P(H) = \emptyset$ and $|P(u', v')| \geq D(H) + 1$.

Proof: Without loss of generality let $y_1 \in V(B)$. Let u' be the last vertex in $V(P[u, v))$ such that $u'y_1 \in E(G)$. Since $vy_2 \in E$, choose v' to be the first vertex in $V(P(u', v])$ such that $v'y_2' \in E(G)$ for some $y_2' \in V(H) - \{y_1\}$. Since $P(u, v)$ is a good segment, $P(u', v')$ is a good segment such that $P(u', v') \subseteq P(u, v)$ and $V(P(u', v')) \cap N(H) = \emptyset$. Since P is maximal and $D(B_1) = D(H)$, this implies that $|P(u', v')| \geq D(B_1) + 1 = D(H) + 1$. □

Lemma 7. If G is $(k+1)$-connected and $|P| < \sigma_2(G) - 1$, then for every component $H \subseteq G - P$ there exists a vertex $v \in H$ such that $\mathrm{d}_G(v) < \frac{\sigma_2(G)}{2}$.

Proof: Let $L = \frac{\sigma_2(G)}{2}$. Suppose that there exists a component $H \subseteq G - P$ such that for every vertex $v \in H$, $\mathrm{d}_G(v) \geq L$. Since $|P| < 2L - 1$ and by Lemma 2 we may assume $|B_1| \geq 2$. We first claim that $|B_1| \geq 3$.

Otherwise, assume $|B_1| = 2$. Then for any vertex z in B_1 we have $|N_P(z)| = d_G(z) - 1 \geq L - 1$. Hence, we have $|N_P(z)| = L - 1$ and $|N_P(H)| = |N_P(z)| \geq k + 1$ as G is $(k+1)$-connected. By Lemma 2, we know for any i, $|P_i| \geq 1$ where $1 \leq i \leq t - 1$. Since $|S| = k$ and the longest requisite path P begins at v_1 and terminates at v_k where v_1 and v_k are the first and last vertices of the ordered set S respectively, there exists i_1 and i_2 such that $V(P_{i_1}) \cap S = \emptyset$ and $V(P_{i_2}) \cap S = \emptyset$. Thus $|P_{i_1}| \geq 2$ and $|P_{i_2}| \geq 2$. Since for any $i \neq i_1, i_2$ with $1 \leq i \leq L - 2$ we have $|P_i| \geq 1$. Thus $|P| \geq L - 1 + L - 2 + 2 = 2L - 1$; a contradiction. Hence $|B_1| \geq 3$.

Next we will show the following claim which is useful in our later proofs.

Claim 1: If $|P| < 2L-1$, then there exists $1 \le i_1 < i_2 < \ldots < i_{k+1} \le t$ such that

(i) $|\{x_{i_1}, \ldots, x_{i_{k+1}}\} \cap N_P(B)| \ge k$

(ii) For any $i_p \ne i_s$, there exist distinct vertices $v, v' \in V(H)$ such that $x_{i_p}v \in E(G)$, $x_{i_s}v' \in E(G)$, and $\{v, v'\} \cap V(B) \ne \emptyset$, that is $|vP_Hv'| \ge D(H)+1$.

In fact since $|B_1| \ge 3$, we have $|B| \ge 2$. Define $X = \{x_i : d_B(x_i) \ge 2, x_i \in V(P)\}$. Then we have $X \subseteq N_P(B)$ and $\sum_{y \in V(B)} d_{V(P)-X}(y) = |N_{V(P)-X}(B)| \le t - |X|$. Hence

$$\begin{aligned}
|B|L &\le \sum_{y \in V(B)} d_G(y) \\
&\le \sum_{y \in V(B)} (d_{B_1}(y) + d_X(y)) + (t - |X|) \\
&\le |B|(|B| + |X|) + t - |X|.
\end{aligned}$$

If $|X| \ge k$, since G is $(k+1)$-connected, Claim 1 holds by the definition of X. Assume $|X| \le k-1$ and set $r = k+1-|X|$ $(r \ge 2)$. Notice that by Lemma 2, $k+1 \le t < L$ which implies $t - L \le -1$. From the above inequality, we have

$$\begin{aligned}
(|B|-1)L &\le |B|(|B|+|X|) + t - |X| - L \\
&= |B|(|B|+|X|+1) - |B| - |X| + t - L \\
&\le |B|(|B|+|X|+1) - (|B|+|X|+1) \\
&= (|B|-1)(|B|+|X|+1).
\end{aligned}$$

Since $|B| \ge 2$, we obtain $(|B_1| + |X|) \ge L \ge k+2$ which implies that $|B_1| \ge k+2-|X| = r+1$. Since $|P| - |X| \ge 2t-1-|X| > t - |X| \ge k+1-|X| = r$ and $G - (X \cup \{c_1\})$ is $(r-1)$-connected, there exist distinct vertices $z_1, z_2, \ldots, z_{r-1}$ in $V(B)$ and $y_1, y_2, \ldots, y_{r-1}$ in $V(P) - X$ such that $z_iy_i \in E(G)$ $(1 \le i \le r-1)$. Set $Y = \{y_1, y_2, \ldots, y_{r-1}\}$ and $Z = \{z_1, z_2, \ldots, z_{r-1}\}$. Notice that $|X \cup Y| = k+1-r+r-1 = k$. Similarly, $|X \cup Z| = k$. Because G is $(k+1)$-connected, there exists some $z_r \in V(H) - Z$ and $y_r \in V(P) - (X \cup Y)$ such that $z_ry_r \in E(G)$. By the definition of X, we can easily check that $X \cup Y \cup \{y_r\}$ is a set that satisfies (i) and (ii).

Now we turn to prove Lemma 7. By Lemma 5, there exists $v \in V(H)$ such that $d_H(v) \le D(H)$. Since G is $(k+1)$-connected, $|S| = k$, and P is a $v_1 - v_k$ path which has at least two good segments. Thus, by

Claim 1 and Lemmas 5 and 6, we have $|P| \geq 2|N_P(v)| - 1 + 2(D(H)) \geq 2d_P(v) - 1 + 2d_H(v) \geq 2L - 1$; a contradiction. □

Lemma 8. If $|P| < \sigma_2(G) - 1$, there is only one component H of $G - P$.

Proof: Suppose there are at least two components $H, H' \subseteq G - P$. Then by Lemma 7, we can take $v \in H$ such that $\mathrm{d}(v) < \frac{\sigma_2(G)}{2}$ and $v' \in H'$ such that $\mathrm{d}(v') < \frac{\sigma_2(G)}{2}$. Since v and v' are nonadjacent, this implies $\sigma_2(G) \leq \mathrm{d}(v) + \mathrm{d}(v') < \sigma_2(G)$, a contradiction. □

So for every longest requisite path P, if $|P| < \sigma_2(G) - 1$, then $G - P$ has only one component, H. Recall $N(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P and $P_i = P(x_i, x_{i+1})$ where $1 \leq i \leq t - 1$. The following corollary is a consequence of Lemmas 7 and 8 and the definition of $\sigma_2(G)$.

Corollary 2. For any $v \in V(P) - N_P(H)$, $d_{G-P}(v) = 0$ and $d_G(v) > \frac{\sigma_2(G)}{2}$.

Lemma 9. If $|P| < \sigma_2(G) - 1$, then for every good segment P_i where $1 \leq i \leq t - 1$, there exist at least two noninsertible vertices in P_i.

Proof: Suppose there is a good segment P_i with no noninsertible vertices, then we can find a requisite path $P' = P[v_1, x_i)x_iP_Hx_{i+1}P(x_{i+1}, v_k]$. Since every vertex in P_i can be inserted into P' by Lemma 3, we can find a requisite path that is longer than P, contradicting the choice of P.

Suppose there exists some good segment, P_i with only one noninsertible vertex v. Then applying Lemmas 3 and 4 we can insert all vertices in $V(P_i) - \{v\}$ into $V(P - P_i)$ to get a requisite path P' including all vertices in $V(P) - \{v\}$ and at least one vertex in H. Thus $|P'| \geq |P|$. Since v is the only noninsertible vertex in P_i and by Corollary 2, $d_{G-P}(v) = 0$, v is itself a component of $G - P'$. And since $d_G(v) > \frac{\sigma_2(G)}{2}$, this component contains no vertex y such that $d(y) < \frac{\sigma_2(G)}{2}$ which contradicts Lemma 7. □

3. Proof of Theorem 8

By contradiction, assume that $p_o(G) < \min\{n - 1, \sigma_2(G) - 2\}$. Let P be a longest requisite path for some ordered set $S = \{v_1, v_2, \ldots, v_k\}$ with $|P| < \min\{n, \sigma_2(G) - 1\}$. By Lemma 8 there exists exactly one component $H \subseteq G - P$. Let $N_P(H) = \{x_1, x_2, \ldots, x_t\}$ in order along P. Since G is $(k+1)$-connected we know that P contains at least two good segments P_i and P_j with $i < j$ such that there exist distinct vertices $w_q, w'_q \in V(H)$ $(q = i, j)$ with $x_qw_q \in E(G)$, $x_{q+1}w'_q \in E(G)$ and $|w_qP_Hw'_q| \geq D(H) + 1$ by Claim 1, and $V(P_q) \cap N_P(H) = \emptyset$ and $|P_q| \geq D(H) + 1$ by Lemma 6 . Then by Corollary 2, for any vertex $x \in V(P_i) \cup V(P_j)$, we have that $d_{G-P}(x) =$

0. By Lemma 9, we know that there are at least two noninsertible vertices in P_i and P_j respectively. Choose u_q and u'_q $(q = i, j)$ to be the first and last noninsertible vertices in P_q, respectively. Notice that a path containing the vertex set $V(P) - (V(P_i) \cup V(P_j))$ in the same order as that in P is a requisite path. We consider the following two cases:

Case 1: $N(u_i) \cap V(P_j) = \emptyset$ or $N(u'_i) \cap V(P_j) = \emptyset$.

Since u_i is noninsertible and $N(u_i) \subseteq V(P)$, by applying Lemma 1 to $P - (P_i \cup P_j)$, we have $d_{P-(P_i \cup P_j)}(u_i) \leq \frac{|P-(P_i \cup P_j)|+3}{2}$ and $d_{P_i}(u_i) \leq |P_i| - 1$. As $d_{P_j}(u_i) = 0$, $d(u_i) = d_{P-(P_i \cup P_j)}(u_i) + d_{P_i}(u_i)$. Choose $h \in V(H)$ such that $d_H(h) \leq D(H)$. Then by Lemmas 1 and 2, $d_P(h) \leq \frac{|P-(P_i \cup P_j)|+3}{2}$. Now

$$\begin{aligned} \sigma_2(G) &\leq d(u_i) + d(h) \\ &\leq \frac{|P-(P_i \cup P_j)|+3}{2} + |P_i| - 1 + \frac{|P-(P_i \cup P_j)|+3}{2} + D(H) \\ &= |P - (P_i \cup P_j)| + |P_i| + D(H) + 2. \end{aligned}$$

Since $|P_j| \geq D(H) + 1$, $\sigma_2(G) \leq |P| + 1$; a contradiction. Thus $N(u_i) \cap V(P_j) \neq \emptyset$. Symmetrically, $N(u'_i) \cap V(P_j) \neq \emptyset$.

Case 2: $N(u_i) \cap V(P_j) \neq \emptyset$ and $N(u'_i) \cap V(P_j) \neq \emptyset$.

Case 2.1 There are some $z \neq w$ in $V(P_j)$ such that $N(z) \cap V(P(x_i, u_i]) \neq \emptyset$ and $N(w) \cap V(P[u'_i, x_{i+1})) \neq \emptyset$.

Choose such z and w with $|P(z, w)|$ as small as possible and, subject to that, choose $b_i \in N_P(z) \cap P(x_i, u_i]$ and $b'_i \in N_P(w) \cap P[u'_i, x_{i+1})$ such that $|P(x_i, b_i]| + |P[b'_i, x_{i+1})|$ is as small as possible.

Define $P' = P[v_1, x_i)x_i P_H x_{i+1} P(x_{i+1}, z) z b_i P(b_i, b'_i) b'_i w P(w, v_k]$ when $z \in P(x_j, w)$ or $P' = P[v_1, x_i)x_i P_H x_{i+1} P(x_{i+1}, w) w b'_i \overline{P}(b'_i, b_i) b_i z P(z, v_k]$ when $w \in P(x_j, z)$, see Figure 5.

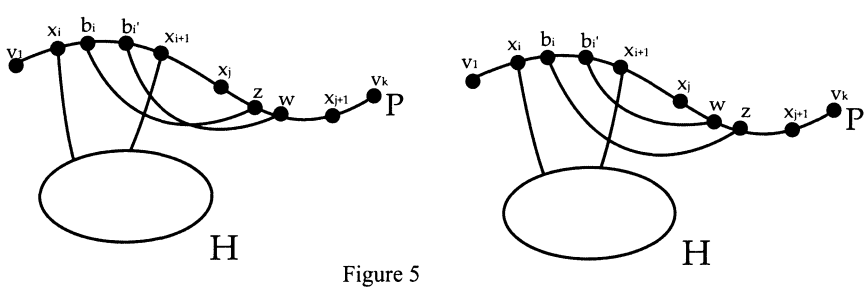

Figure 5

It is easy to check that P' is a requisite path. By the choices of w, z, b_i, b'_i and Lemmas 3 and 4, $N(u_i) \cap V(P(w, z)) = \emptyset$ and all vertices in $V(P(x_i, b_i)) \cup V(P(b'_i, x_{i+1}))$ can be inserted into P'. Thus by the maximality of P, we can conclude $|P(w, z)| \geq D(H) + 1$ (or $|P(z, w)| \geq D(H) + 1$).

As in Case 1, we choose $h \in V(H)$ such that $d_H(h) \leq D(H)$. Then

$$\begin{aligned} \sigma_2(G) &\leq d(u_i) + d(h) \\ &\leq \tfrac{|P-(P_i \cup P(w,z)|+3}{2} + |P_i| - 1 + \tfrac{|P-(P_i \cup P(w,z))|+3}{2} + D(H) \\ &= |P - (P_i \cup P(w,z))| + |P_i| + D(H) + 2. \end{aligned}$$

Thus, $\sigma_2(G) \leq |P| + 1$; a contradiction.

Case 2.2 There is only one vertex, say z, in $V(P_j)$ such that $zu_i \in E(G)$, $zu_i' \in E(G)$ and $N(y) \cap (V(P_j) - \{z\}) = \emptyset$ for every $y \in V(P(x_i, u_i)) \cup V(P(u_i', x_{i+1}))$.

Symmetrically, we may assume that there is only one vertex, say m, in $V(P_i)$ such that $mu_j \in E(G)$, $mu_j' \in E(G)$ and $N(y') \cap (V(P_i) - \{m\}) = \emptyset$ for every $y' \in V(P(x_j, u_j)) \cup V(P(u_j', x_{j+1}))$.

If $|P_j| \geq D(H) + 2$, then, ignoring z, we still have $D(H) + 1$ available vertices all of which are not adjacent to both v_i and u_i'. By using the same method as that in Case 1, we can get $|P| \geq \sigma_2(G) - 1$, a contradiction.

If $|P_j| = D(H) + 1$, symmetrically, we may assume that $|P_i| = D(H) + 1$. This implies that u_i and u_i' are the first and last vertices in P_i, respectively. Similarly, u_j and u_j' are the first and last vertices in P_j, respectively. Consider $P_1' = P[v_1, u_i)u_i z P(z, u_j')u_j' m P(m, x_j)x_j P_H x_{j+1} P(x_{j+1}, v_k]$ and $P_2' = P[v_1, m)mu_j P(u_j, z)zu_i' P(u_i', x_j)x_j P_H x_{j+1} P(x_{j+1}, v_k]$. See Figure 6. Clearly, P_1' and P_2' are both requisite paths. From P_1', we see that $|P(u_i, m)| + |P[u_j, z)| \geq D(H) + 1$, otherwise $|P_1'| > |P|$. From P_2', we see that $|P(m, u_i')| + |P(z, u_j']| \geq D(H) + 1$, otherwise $|P_2'| > |P|$. Including m and z in our calculation, we have that $|P_i| + |P_j| \geq 2D(H) + 2 + 2 = 2D(H) + 4$ which contradicts the fact that $|P_i| = |P_j| = D(H) + 1$. □

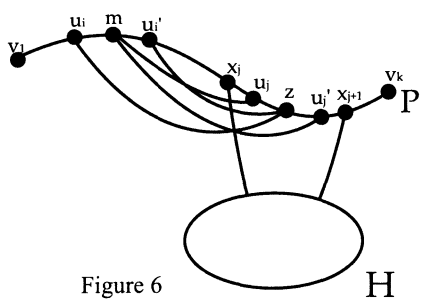

Figure 6

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