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# The s-degenerate r-Lah numbers

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**Abstract**

Recently, Belbachir and Belkhir gave some recurrence relations for the $r$-Lah numbers. In this paper, we give other properties for the $r$-Lah numbers, we introduce and study a restricted class of these numbers.

**Keywords.** $r$-Lah numbers; $s$-degenerate $r$-Lah numbers; recurrence relations.
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## 1 Introduction

For $r \in \mathbb{N}$, the $r$-Lah number $\left\lfloor {n \atop k} \right\rfloor_r$ play an important role in combinatorics and counts the number of partitions of a $n$-set into $k$ ordered blocks such that the $r$ first elements are in different blocks. Recall that the exponential generating function of $r$-Lah numbers is to be

$$(1) \qquad \sum_{n\geq k} \left\lfloor {n+r \atop k+r} \right\rfloor_r \frac{t^n}{n!} = \frac{1}{k!}\frac{t^k}{(1-t)^{k+2r}}.$$

These numbers satisfy the following triangular recurrence relation

$$(2) \qquad \left\lfloor {n \atop k} \right\rfloor_r = \left\lfloor {n-1 \atop k-1} \right\rfloor_r + (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor_r, \quad n \geq k \geq r.$$

and have the explicit expression

$$(3) \qquad \left\lfloor {n \atop k} \right\rfloor_r = \frac{(n-r)!}{(k-r)!}\binom{n+r-1}{k+r-1} = \frac{(n+r-1)!}{(k+r-1)!}\binom{n-r}{k-r}, \quad n \geq k \geq r.$$

Recently, Belbachir and Belkhir gave some recurrence relations for the $r$-Lah numbers, see also [3]. In this paper, we give other properties for these numbers and we study a restricted class of them by giving their generating function and some combinatorial recurrence relations.

**Definition 1** *Let $s$ be a positive integer and $r$ be a non-negative integer. The $s$-degenerate $r$-Lah numbers, denoted by $\left\lfloor {n \atop k} \right\rfloor_r^{s\uparrow}$, count the number of partitions of a $n$-set into $k$ ordered blocks, each one has a length at least $s$ elements and such that the first $r$ elements must be in different blocks.*

## 2 Some properties for the $r$-Lah numbers

Two recurrence relations for the $r$-Lah numbers are given by the following proposition on using in (3) the fact that $\binom{n+r-1}{k+r-1} = \binom{n+r-2}{k+r-1} + \binom{n+r-2}{k+r-2}$ and $\binom{n-r}{k-r} = \binom{n-r-1}{k-r} + \binom{n-r-1}{k-r-1}$.

**Proposition 2** *For $r < k$ we have*

$$\begin{aligned} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r &= (n-r) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r + \frac{n-r}{k-r} \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r, \\ \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r &= (n+r-1) \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r + \frac{n+r-1}{k+r-1} \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rfloor_r. \end{aligned}$$

Note that for $r = 0$ or 1 we obtain Corollary 2.1 given in [2].

Now, it is known from [1] that the polynomial $P_n(x) = \sum_{k=0}^{n} \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_0 x^k$ has only real roots, distinct and non-positive. The following proposition shows a similar result for the $r$-Lah numbers.

**Proposition 3** *The polynomial $P_n(x;r) := \sum_{k=0}^{n} \left\lfloor \begin{smallmatrix} n+r \\ k+r \end{smallmatrix} \right\rfloor_r x^k$ has only non-positive real roots.*

**Proof.** Set $T_n(x;r) := x^n \exp(x) P_n(x;r)$. The recurrence relation (2) shows that $T_n(x;r) = x^2 \frac{d}{dx} T_{n-1}(x;r)$. So, by induction on $n$, the function $T_n(x;r)$ vanishes on $]-\infty, 0]$. ■

On using Newton's inequality given by Hardy et al. [4, p. 52] we may state that the sequence $\left( \left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rfloor_r ; 0 \leq k \leq n \right)$ is strongly log-concave.

Similarly to a result of Ahuja and Enneking [1] on the Lah numbers, on using (2), one can prove by induction on $n$, the following proposition.

**Proposition 4** *We have*

$$D^n \left( \frac{1}{x^{2r}} \exp\left(\frac{1}{x}\right) \right) = \frac{(-1)^n}{x^{n+2r}} \exp\left(\frac{1}{x}\right) \sum_{k=0}^{n} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r \frac{1}{x^k}.$$

## 3 The $s$-degenerate $r$-Lah numbers

From the definition, the $s$-degenerate $r$-Lah numbers are zero when $n < sk$ or $k < r$. We start this section by giving the following theorem.

**Theorem 5** *For $n > sk$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} = \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n-sk} (n_1+s)\cdots(n_r+s).$$

**Proof.** To partition the set $[n] := \{1, 2, \ldots, n\}$ into $k$ ordered blocks such that each block has a length $\geq s$ and the elements of the set $[r]$ are in different blocks, we construct $k$ subsets $B_1, \ldots, B_k$ for which we insert the elements of the set $[r]$ be in $B_1, \ldots, B_r$ (one element in each subset). To construct the $k$ subsets, there is $\frac{1}{(k-r)!}\binom{n-r}{n_1,\ldots,n_k}$ ways to choose $n_1, \ldots, n_k$ in $\{r+1, \ldots, n\}$ such that $n_i + 1 \geq s$, $n_i$ elements are in $B_i$ for $i = 1, \ldots, r$, and, $n_i \geq s$, $n_i$ elements are in $B_i$ for $i = r+1, \ldots, k$.
Now, to form the $k$ ordered blocks, we permute the elements of each subset. The number of permutations is equal to

$$(|B_1|!) \cdots (|B_r|!) (|B_{r+1}|!) \cdots (|B_k|!) = (n_1 + 1)! \cdots (n_r + 1)! n_{r+1}! \cdots n_k!.$$

By meaning $M$ for the set of the vectors $(n_1, \ldots, n_k)$ such that

$$n_1 + \cdots + n_k = n - r, \ (n_1 + 1, \ldots, n_r + 1, n_{r+1}, \ldots, n_k) \in \{s, s+1, \ldots\}^k,$$

the total number of such partitions is

$$\begin{aligned}
\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{s\uparrow} &= \frac{1}{(k-r)!} \sum_{(n_1,\ldots,n_k)\in M} \binom{n-r}{n_1,\ldots,n_k} \prod_{i=1}^{r} (n_i+1)! \prod_{i=r+1}^{k} n_i! \\
&= \frac{(n-r)!}{(k-r)!} \sum_{(n_1,\ldots,n_k)\in M} (n_1+1)\cdots(n_r+1) \\
&= \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n-sk} (n_1+s)\cdots(n_r+s)
\end{aligned}$$

which complete the proof. ■
As a consequence of Theorem 5, we may state:

**Corollary 6** *The $s$-degenerate $r$-Lah numbers have generating function*

$$\sum_{n\geq k} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow} \frac{t^n}{n!} = \frac{1}{k!} \frac{t^{sk+(s-1)r}}{(1-t)^{k+2r}} \left(1 + (s-1)(1-t)\right)^r.$$

**Proof.** For $|t| < 1$, Theorem 5 shows that we have

$$\begin{aligned}
&\sum_{n\geq k} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{s\uparrow} \frac{t^n}{n!} \\
= \ & \frac{1}{k!} \sum_{n\geq sk+(s-1)r} t^n \sum_{(n_1+s)+\cdots+(n_{k+r}+s)=n+r} (n_1+s)\cdots(n_r+s) \\
= \ & \frac{1}{k!} \sum_{n\geq sk+(s-1)r} t^n \sum_{j_1+\cdots+j_{k+r}=n+r \ \ j_1\geq s,\ldots,j_{k+r}\geq s} j_1\cdots j_r \\
= \ & \frac{1}{k!} \sum_{n\geq sk+(s-1)r} \ \sum_{j_1+\cdots+j_{k+r}=n+r, \ \ j_1\geq s,\ldots,j_{k+r}\geq s} j_1\cdots j_r t^{j_1+\cdots+j_{k+r}-r}
\end{aligned}$$

and this is equal to

$$\begin{aligned}
&\frac{1}{k!}\sum_{j_1\geq s,\ldots,j_{k+r}\geq s}\left(j_1t^{j_1-1}\right)\cdots\left(j_rt_r^{j_r-1}\right)\left(t^{j_{1+r}}\right)\cdots\left(t^{j_{k+r}}\right)\\
=\;&\frac{1}{k!}\left(\sum_{j\geq s}jt^{j-1}\right)^r\left(\sum_{j\geq s}t^j\right)^k\\
=\;&\frac{1}{k!}\left(\frac{d}{dt}\frac{t^s}{1-t}\right)^r\left(\frac{t^s}{1-t}\right)^k\\
=\;&\frac{1}{k!}\frac{t^{sk+(s-1)r}}{(1-t)^{k+2r}}\left(1+(s-1)(1-t)\right)^r
\end{aligned}$$

which complete the proof. ■

**Corollary 7** *For $r\leq k$ we have*

$$\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor_r^{s\uparrow}=\frac{(n-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\binom{n+j-(s-1)k-1}{k+j-1}(s-1)^{r-j}$$

*and satisfy for $r<k$ the recurrence relations*

$$\begin{aligned}
\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor_r^{s\uparrow}&=(n-r)\left\lfloor\begin{matrix}n-1\\k\end{matrix}\right\rfloor_r^{s\uparrow}+\frac{s!}{k-r}\binom{n-r}{s}\left\lfloor\begin{matrix}n-s\\k-1\end{matrix}\right\rfloor_r^{s\uparrow},\\
\frac{1}{k-r}\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor_{r+1}^{s\uparrow}&=\frac{s-1}{n-r}\left\lfloor\begin{matrix}n\\k\end{matrix}\right\rfloor_r^{s\uparrow}+\frac{(n-r-1)!\,(k-r+1)}{(n+s-r)!}\left\lfloor\begin{matrix}n+s\\k+1\end{matrix}\right\rfloor_r^{s\uparrow}.
\end{aligned}$$

**Proof.** On using Corollary 6, we get

$$\sum_{n\geq sk+(s-1)r}\left\lfloor\begin{matrix}n+r\\k+r\end{matrix}\right\rfloor_r^{s\uparrow}\frac{t^n}{n!}=\sum_{j=0}^{r}\binom{r}{j}\frac{t^{(s-1)(k+r)+j}}{k!}\frac{(s-1)^j\,t^{k-j}}{(1-t)^{k-j+2r}},$$

and on using (1) and (3) to express the factor $\frac{t^{k-j}}{(1-t)^{k-j+2r}}$ as

$$(k-j)!\sum_{n\geq k-j}\left\lfloor\begin{matrix}n+r\\k-j+r\end{matrix}\right\rfloor_r\frac{t^n}{n!}=\sum_{n\geq k-j}\binom{n+2r-1}{k-j+2r-1}t^n,$$

the first identity follows by identification. The recurrence relations can be obtained by applying the identity $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ to the coefficients $\binom{n+j-(s-1)k-1}{k+j-1}$ and $\binom{r}{j}$ in the first identity. ■

**Remark 8** *From Corollary 6 and (1) we get*

$$\sum_{n\geq k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_r^{s\uparrow}\frac{t^n}{n!}=(s-(s-1)\,t)^r\,t^{(s-1)(k+r)}\sum_{n\geq k}\begin{bmatrix}n+r\\k+r\end{bmatrix}_r\frac{t^n}{n!}$$

*which proves that we have*

$$\begin{bmatrix}N\\k\end{bmatrix}_r^{s\uparrow}=\frac{(N-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\binom{n+j-1}{k+r-1}s^{r-j}\,(1-s)^j\,,\ N=n+(s-1)\,k.$$

The $s$-degenerate $r$-Lah numbers possess a probabilistic interpretation given by the following proposition.

**Proposition 9** *Let $\lambda$ be a real number with $0<\lambda<1$, $\{X_n\}$ and $\{Y_n\}$ two independent sequences of independent random variables with*

$$P\,(X_n=j)=\lambda\,(1-\lambda)^j\,,\ P\,(Y_n=j)=\frac{\lambda^2\,(j+s)}{1+(s-1)\,\lambda}\,(1-\lambda)^j\,,\quad j\geq 0,$$

*and let $S_{k,r}=X_1+\cdots+X_k+Y_1+\cdots+Y_r$. Then, we have*

$$P\,(S_{k,r}=n)=\frac{\lambda^{k+2r}\,(1-\lambda)^n}{(1+(s-1)\,\lambda)^r}\frac{k!}{(n+sk+(s-1)\,r)!}\begin{bmatrix}n+s\,(k+r)\\k+r\end{bmatrix}_r^{s\uparrow}.$$

**Proof.** The proof follows from the moment's generating function of $S_{k,r}$ given by $\mathrm{E}\left(t^{S_{k,r}}\right)=\left(\mathrm{E}\left(t^{X_1}\right)\right)^k\left(\mathrm{E}\left(t^{Y_1}\right)\right)^r$. ∎

# 4 Combinatorial recurrence relations

In this section, we establish some combinatorial recurrence relations.

**Proposition 10** *For $n>sk\geq sr\geq 1$ we have*

$$\begin{aligned}\begin{bmatrix}n\\k\end{bmatrix}_r^{s\uparrow} &= (n+k-1)\begin{bmatrix}n-1\\k\end{bmatrix}_r^{s\uparrow}+s!\binom{n-r-1}{s-1}\begin{bmatrix}n-s\\k-1\end{bmatrix}_r^{s\uparrow}\\ &\quad +s!r\binom{n-r-1}{s-2}\begin{bmatrix}n-s\\k-1\end{bmatrix}_{r-1}^{s\uparrow}.\end{aligned}$$

**Proof.** To partition the set $[n]$ into $k$ ordered blocks such that each block has a length $\geq s$ and the first $r$ elements are in different blocks, we separate the element $n$ and proceed as follows: **(a)** if the element $n$ is in a block of length $\geq s+1$, there is $\left[\begin{smallmatrix}n-1\\k\end{smallmatrix}\right]_r^{s\uparrow}$ ways to partition the set $[n-1]$ into $k$ ordered blocks such that each block has a length $\geq s$ and the first $r$ elements are in different blocks. The element $n$ (not really used) can be inserted in the $k$ blocks with $n-1+k$ ways because it can be inserted with $l+1$ ways in the ordered block $\{a_1,\ldots,a_l\}$ after $a_i$ $(i=1,\ldots,l)$ or before $a_1$. So, we count in this case $(n-1+k)\left\lfloor{n-1 \atop k}\right\rfloor_r^{s\uparrow}$ ways. **(b)** if the element $n$ is in a block of length $s$ and this block no contains any element of the first $r$ elements, there is $\binom{n-1-r}{s-1}$ ways to choose $s-1$ elements to be with this element in the same subset, $s!$ ways to permute the elements of this subset and the remaining $n-s$ elements can be partitioned into $k-1$ ordered blocks (such that each block has a length $\geq s$ and the first $r$ elements must be in different blocks) in $\left\lfloor{n-s \atop k-1}\right\rfloor_r^{s\uparrow}$ ways. The number of ways in this case is $s!\binom{n-1-r}{s-1}\left\lfloor{n-s \atop k-1}\right\rfloor_r^{s\uparrow}$. **(c)** if the element $n$ is in a block of length $s$ and this block contains one element of the first $r$ elements (this case is considered only when $s\geq 2$), there is $\binom{n-1-r}{s-2}$ ways to choose $s-2$ elements to be with this element in the same subset, $r$ ways to choose an element between the first $r$ elements to be with the element $n$ in the same subset, $s!$ ways to permute the elements of this subset and the remaining $n-s$ elements can be partitioned into $k-1$ ordered blocks (such that each block has a length $\geq s$ and the first $r-1$ elements must be in different blocks) in $\left\lfloor{n-s \atop k-1}\right\rfloor_{r-1}^{s\uparrow}$ ways. The number of ways in this case is $s!r\binom{n-r-1}{s-2}\left\lfloor{n-s \atop k-1}\right\rfloor_{r-1}^{s\uparrow}$. So, the number of all partitions is $\left\lfloor{n \atop k}\right\rfloor_r^{s\uparrow}=(n+k-1)\left\lfloor{n-1 \atop k}\right\rfloor_r^{s\uparrow}+s!\binom{n-1-r}{s-1}\left\lfloor{n-s \atop k-1}\right\rfloor_r^{s\uparrow}+s!r\binom{n-r-1}{s-2}\left\lfloor{n-s \atop k-1}\right\rfloor_{r-1}^{s\uparrow}$. ■

**Proposition 11** *We have*

$$\left\lfloor{n+r \atop k+r}\right\rfloor_r^{s\uparrow}=\sum_{j\geq 0}\binom{n}{j}\left\lfloor{n-j \atop k}\right\rfloor_0^{s\uparrow}\left\lfloor{j+r \atop r}\right\rfloor_r^{s\uparrow}.$$

**Proof.** For fixed $j$, there is $\binom{n}{n-j}\left\lfloor{n-j \atop k}\right\rfloor_0^{s\uparrow}$ ways to form the $k$ ordered blocks (each one is of length $\geq s$) with using only the remaining $n-j$ elements of the set $\{r+1,\ldots,n+r\}$. To form the $r$ remaining blocks, there is $\left\lfloor{j+r \atop r}\right\rfloor_r^{s\uparrow}$ ways to form $r$ ordered blocks (each one is of length $\geq s$) on using the $j$ elements of the set $\{r+1,\ldots,n+r\}$ and the elements of the set $[r]$ (not really used) such that each block contains one element of the set $[r]$. Then, the number of ordered partitions is given by $\left\lfloor{n+r \atop k+r}\right\rfloor_r^{s\uparrow}=\sum_{j\geq 0}\binom{n}{n-j}\left\lfloor{n-j \atop k}\right\rfloor_0^{s\uparrow}\left\lfloor{j+r \atop r}\right\rfloor_r^{s\uparrow}$. ■

**Proposition 12** *For $n \geq sk \geq sr \geq 1$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} = \sum_{j \geq s} j! \binom{n-r}{j-1} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}.$$

**Proof.** To partition the set $[n]$ into $k$ ordered blocks such that each block has a length $\geq s$ and the first $r$ elements are in different blocks, the element $r$ can be in a block of length $j \geq s$ in $j!\binom{n-r}{j-1}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}$ ways: **(a)** $\binom{n-r}{j-1}$ is the number of ways for choosing $j-1$ elements between $(n-1)-(r-1)$ elements (none is of the first $r-1$ elements) to be in the same subset with the element $r$ **(b)** $j!$ is the number of ways to permute the elements of this subset and **(c)** $\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}$ is the number of ways to partition the remaining $n-j$ elements into $k-1$ blocks such that each block has a length $\geq s$ and the first $r-1$ elements are in different blocks. ■

For $s = 1$ in Proposition 12 we get Theorem 3 given in [2].

**Proposition 13** *For $r \geq 1$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} - \begin{bmatrix} n \\ k \end{bmatrix}_{r+1}^{s\uparrow} = r \sum_{j \geq s} j! \binom{n-r-1}{j-2} \begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}.$$

**Proof.** The number of partitions of the set $[n]$ into $k$ ordered blocks such that each block has a length $\geq s$ and the first $r$ elements are in different blocks but not the element $r+1$ there is $\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} - \begin{bmatrix} n \\ k \end{bmatrix}_{r+1}^{s\uparrow}$ ways. This number can be obtained by considering the element $r+1$ to be in one of the $r$ subsets which contain the first $r$ elements. The element $r+1$ can be with the element $i$, $(i = 1, ..., r)$, in a subset of cardinality $j$ in $j!r\binom{n-r-1}{j-2}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}$ ways: **(a)** $\binom{n-r-1}{j-2}$ is the number of ways to choose $j-2$ elements between the last $n-r-1$ elements to be in the same subset with $i$ and $r+1$, **(b)** $j!$ is the number of ways to permute the elements of this subset, **(c)** $\begin{bmatrix} n-2-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}$ is the number of ways to partition the remaining $n-j$ into $k-1$ ordered blocks such that each block has a length $\geq s$ and the elements of the set $[r]-\{i\}$ are in different blocks and **(d)** $r$ is the number of ways to choose $i$ between the first $r$ elements to be with the element $r+1$ in the same block. Then, the number of all partitions is $\begin{bmatrix} n \\ k \end{bmatrix}_r^{s\uparrow} - \begin{bmatrix} n \\ k \end{bmatrix}_{r+1}^{s\uparrow} = r\sum_{j \geq s} j!\binom{n-r-1}{j-2}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}^{s\uparrow}$. ■

We note that for $s = 1$, Theorem 5 gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_r := \begin{bmatrix} n \\ k \end{bmatrix}_r^{1\uparrow} = \frac{(n-r)!}{(k-r)!} \sum_{n_1+\cdots+n_k=n-k} (n_1+1)\cdots(n_r+1),$$

Corollary 7 gives (3) and Proposition 13 gives

$$\begin{bmatrix} n \\ k \end{bmatrix}_r - \begin{bmatrix} n \\ k \end{bmatrix}_{r+1} = r\sum_{j\geq 1} j!\binom{n-r-1}{j-2}\begin{bmatrix} n-j \\ k-1 \end{bmatrix}_{r-1}.$$

Also, for $r = 0$ in Proposition 10 and Corollary 6, the $s$-degenerate Lah numbers $\left\lfloor{n \atop k}\right\rfloor^{s\uparrow} := \left\lfloor{n \atop k}\right\rfloor_0^{s\uparrow}$ satisfy

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}^{s\uparrow} &= (n+k-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}^{s\uparrow} + s!\binom{n-1}{s-1}\begin{bmatrix} n-s \\ k-1 \end{bmatrix}^{s\uparrow}, \\ \begin{bmatrix} n \\ k \end{bmatrix}^{s\uparrow} &= \frac{n!}{k!}\binom{n-1-(s-1)\,k}{k-1}, \quad n \geq sk > 0. \end{aligned}$$

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## References

[1] J. C. Ahuja, E. A. Enneking, Concavity property and a recurrence relation for associated Lah numbers. *The Fibonacci Quarterly* 17 (1979), 158–61.

[2] H. Belbachir, A. Belkhir, Cross recorrence relations for the $r$-Lah numbers. *Ars Combin.* 110 (2013), 199–203.

[3] H. Belbachir, I. E. Bousbaa, Combinatorial identities for the $r$-Lah numbers. *Ars Combin.* 115 (2014), 453–458.

[4] G. H. Hardy, J. E. Littlewood, G. Ploya, Inequalities (Cambridge: The University Press, 1952).

[5] I. Lah, Eine neue art yon zahlen, ihre eigenschaften und anwendung in der mathematischen statistik,Mitteilungsbl, *Math. Statist.* 7 (1955), 203–212.