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SOME NEW METHODS IN RECONSTRUCTION THEORY

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This is an expository paper in which we describe two new methods in the graph reconstruction problem.

The Algebra of Subgraphs deals with relations that hold among the subgraphs of a graph. It is the combinatorial principle used in Tutte's "All the King's Horses".

The method of Partial Automorphisms is concerned with extending an isomorphism between two graphs to an automorphism of some larger graph. We describe its use in connection with pseudo-similar vertices in a graph, and with the problem of reconstructing graphs with only two degrees, k and $k-1$.

I. THE ALGEBRA OF SUBGRAPHS

1. THE RECONSTRUCTION CONJECTURE

Let G be a simple graph with vertex-set $V(G)$ and edge-set $E(G)$. We shall use the notation of Bondy and Murty [4], so that G has $\nu(G)$ vertices and $\varepsilon(G)$ edges.

We begin by defining the graph reconstruction problem. We use $G-v$ to denote the *vertex-deleted* subgraph of G got by deleting the vertex v and all its incident edges.

Definition. *Graphs* G *and* H *are* reconstructions *of each other if there exists a bijection* $\psi : V(G) \rightarrow V(H)$ *such that for every* $v \in V(G)$, $G-v \cong H-\psi(v)$.

1.1 Reconstruction Conjecture. If G and H are reconstructions of each other with at least three vertices, then $G \cong H$.

We refer the reader to Bondy and Hemminger [3] or to Nash-Williams [16] for comprehensive surveys of the reconstruction problem.

From the statement of the reconstruction conjecture, we see that it is not the individual graphs G and H which are important, but rather their isomorphism type. The same is true of the vertex-deleted subgraphs $G-v$ and $H-\psi(v)$. Accordingly, it is convenient to rephrase the reconstruction conjecture as follows.

1.2 RC1. If G has at least three vertices, then the isomorphism type of G is determined by the isomorphism types of its vertex-deleted subgraphs.

We defined the vertex-deleted subgraph $G-v$ by deleting the vertex v. It is worthwhile to look at it from another point of view. It is very easy to see that $G-v$ is really the subgraph of G induced by all vertices but v, i.e.,

$$G-v \equiv G[V(G)-\{v\}].$$

This point of view leads naturally to the Algebra of Subgraphs described in subsequent sections.

There is another, equally fundamental way of forming subgraphs of a graph. We can also consider *edge-induced* subgraphs of G. In particular, if $e \in E(G)$, we define the *edge-deleted* subgraph $G-e$ as that subgraph of G induced by all edges but e:

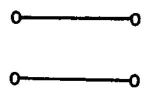
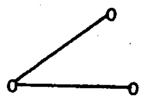
$$G-e \equiv G[E(G)-\{e\}].$$

It is only natural that there should be another reconstruction conjecture based on this way of forming subgraphs.

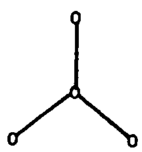
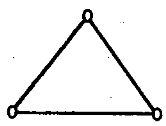
Definition. *Graphs* G *and* H *are* edge-reconstructions *of each other if there exists a bijection* $\psi : E(G) \to E(H)$ *such that for every* $e \in E(G)$, $G-e \simeq H-\psi(e)$.

Two examples of graphs which are edge-reconstructions of each other follow.

1.3 Example.

G =  H = 

1.4 Example.

G =  H = 

Notice that these examples differ slightly from the usual ones in that we have left out any isolated vertices. This is because isolated vertices are lost when an edge-induced subgraph is formed, and so need not be included.

1.5 Edge-Reconstruction Conjecture. If G and H are edge-reconstructions of each other with at least four edges, than $G \simeq H$.

Again we notice that it is really the isomorphism type of G and H which is important, and we rephrase the edge-reconstruction conjecture as follows.

1.6 ERC1. If G has at least four edges then the isomorphism type of G is determined by the isomorphism types of its edge-deleted subgraphs.

We have two reconstruction conjectures because there are two fundamental ways of forming subgraphs of a graph. These two ways of forming subgraphs define two order relations on the set of all graphs.

If g is an edge-induced subgraph of G, we write $g \leq G$.

If g is a vertex-induced subgraph of G, we write $g \leq G$.

It is interesting to rephrase both reconstruction conjectures at once, in terms of these two order relations.

1.7 RC2. If G has at least four edges, then the isomorphism type of G is determined by the isomorphism types of its *maximal proper subgraphs*.

2. TWO TYPES OF SUBGRAPH

Corresponding to the two kinds of subgraph, we define the following notation.

Let g and G be graphs. We define

$$s(g,G) = |\{V' \subseteq V(G) : G[V'] \simeq g\}|.$$

In words, $s(g,G)$ is the number of vertex-induced subgraphs of G which are isomorphic to g.

If g has no isolated vertices we define

$$n(g,G) = |\{E' \subseteq E(G) : G[E'] \simeq g\}|.$$

In words, $n(g,G)$ is the number of edge-induced subgraphs of G which are isomorphic to g.

It is convenient to extend the definition of $n(g,G)$ to include graphs g with isolated vertices. Let g have k isolated vertices. Write

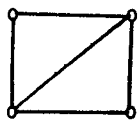
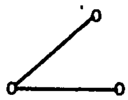
$$g = g' + kK_1,$$

i.e., the disjoint union of g' and k isolated vertices, where g' has no isolated vertices. Define

$$n(g,G) = \binom{\nu(G)-\nu(g')}{k} n(g',G),$$

i.e., we choose a subgraph isomorphic to g' and then we choose k of the remaining vertices, in all possible ways. In case g' is the empty graph $\emptyset$, we define $n(\emptyset,G) = 1$, for any graph G.

2.1 Example.

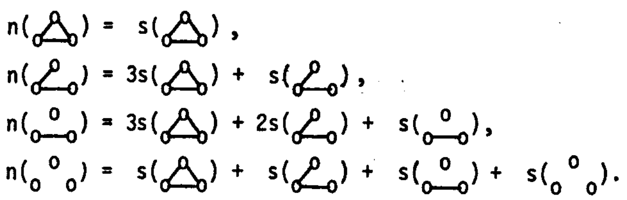
G =  g = 

$s(g,G) = 2$ and $n(g,G) = 8$.

The Algebra of Subgraphs is concerned with relations among the $n(g)$ and $s(g)$, as g ranges over all isomorphism types of graphs. Here, and in the sequel, we suppress the G in $s(g,G)$ and $n(g,G)$, unless the graph G need be specified. Most of the formulae will be true for any G, and as such, do not depend on G.

There are several elementary relations connecting the $s(g)$ and $n(g)$, as illustrated below in the case of three vertices.

2.2 Example.



This is an example of a simple and well-known result (see Biggs [1],[2], Erdös, Lovász, and Spencer [6], and Kocay [11]).

2.3 Theorem. *Let* $g_1, g_2, \ldots, g_m$ *be a list of all isomorphism types of graphs on* n *vertices. Then*

$$n(g_i) = \sum_{j=1}^{m} M_{ij} s(g_j), \qquad i = 1,2,\ldots,m,$$

where $M_{ij} = n(g_i,g_j)$. *The matrix* $M_n = [M_{ij}]_{m\times m}$ *is non-singular, and the entries of* M_n^{-1} *are integers.*

Proof. Let G be any graph, and let $g_i \subseteq G$. Then $G[V(g_i)] \cong g_j$, for some j. $G[V(g_i)]$ contains exactly M_{ij} edge-induced subgraphs isomorphic to g_i. Each of these is a subgraph of G with the same vertex-set $V(g_i)$. Thus the formula holds.

To see that M_n is invertible, order the g_i, $i = 1,2,\ldots,m-1$, so that $\varepsilon(g_i) \geq \varepsilon(g_{i+1})$. Then M_n is a lower triangular integral matrix with a diagonal of ones, as in Example 2.2. So is M_n^{-1}. □

Clearly M_n represents a change of basis transformation in the vector space spanned by the $s(g_i)$. Theorem 2.3 allows us to work with whichever basis, $s(g_i)$ or $n(g_i)$, that we choose.

It was shown by Whitney [19] in 1932 that the number of the $n(g_i)$, $i = 1,2,\ldots,m$, that are algebraically independent is equal to the number of the g_i that are connected.

3. KELLY'S LEMMA AND THE ALGEBRA OF SUBGRAPHS

One of the first results ever proved about reconstruction is the following.

3.1 Kelly's Lemma. *Let* g *and* G *be graphs such that* $\nu(g) < \nu(G)$. *Then*

$$s(g,G) = \frac{1}{\nu(G)-\nu(g)} \sum_{v\in V(G)} s(g,G-v).$$

(The same result holds for $n(g,G)$.)

Proof. If $g \leq G$, then if we delete any vertex $v \in V(G) - V(g)$, we must still have $g \leq G-v$. Therefore the summation above counts each subgraph $g \leq G$ exactly $\nu(G) - \nu(g)$ times.

The same argument proves that this formula also holds for $n(g,G)$. Alternatively, we can apply the matrix $M_{\nu(G)}$ to this linear formula.□

3.2 Kelly's Edge-Lemma. *Let* g *and* G *be graphs with no isolated vertices, such that* $\nu(g) \leq \nu(G)$ *and* $\varepsilon(g) < \varepsilon(G)$. *Then*

$$n(g,G) = \frac{1}{\varepsilon(G)-\varepsilon(g)} \sum_{e\in E(G)} n(g,G-e). \qquad \square$$

In 1.7 we rephrased the reconstruction conjectures 1.1 and 1.5 in terms of maximal proper subgraphs. By 3.1 and 3.2 we can now reformulate them as follows.

3.3 RC3. If G has at least four edges, then the isomorphism type of G is determined by the isomorphism types of its *proper subgraphs*.

This then, is what the reconstruction problem is all about: is a graph determined by its subgraphs?

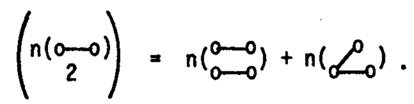
For a long time, Kelly's Lemmas were virtually all that was known about the reconstruction problem. For example, disconnected graphs are determined by their subgraphs essentially because we can use Kelly's Lemma to count the number of each type of maximal connected subgraph, i.e., we can find the connected components.

Trees are determined by their subgraphs, largely for the following reason. Each tree has either a central vertex or a central edge. We can break a tree into its *branches*, that is, its maximal subtrees which have degree one at the centre. We can count the number of each type of maximal branch, using Kelly's Lemma, and then put the tree together again (see Bondy and Hemminger [3], Nash-Williams [16], or Kocay [10]).

The main difficulty with Lemma 3.1 is that we must have $\nu(g) < \nu(G)$; we obtain no information about spanning subgraphs of G. In his famous paper "All the King's Horses" [18], W.T. Tutte succeeded in counting the number of hamiltonian cycles of G, the number of spanning trees of G, and other spanning subgraphs. His method was to use some polynomials associated with a graph. We present here a direct combinatorial approach to count these same subgraphs. The method rests on the following observation.

If we choose two distinct edges of a graph G, either we get independent edges, or we get adjacent edges:

3.4 Example.



This is a special case of the following theorem (see Kocay [11]).

3.5 Edge Theorem. *Let* g_i, $i \geq 1$, *be a list of all isomorphism types of graphs with no isolated vertices. Let* g *and* g' *be any two such graphs. Then*

$$n(g)n(g') \equiv \sum_{i \geq 1} b_i \, n(g_i),$$

where the coefficient b_i *is the number of decompositions of* $E(g_i)$ *into* $E \cup E'$ *(it need not be that* $E \cap E' = \emptyset$*) such that* $g_i[E] \simeq g$ *and* $g_i[E'] \simeq g'$.

Proof. Let G be any graph, and let $g \subseteq G$ and $g' \subseteq G$. Then $G[E(g) \cup E(g')] \simeq g_i$, for some g_i. But $E(g_i)$ has exactly b_i decompositions into $E \cup E'$ such that $g_i[E] \simeq g$ and $g_i[E'] \simeq g'$. Clearly $G[E] \simeq g$ and $G[E'] \simeq g'$ for each such decomposition. □

This theorem has essentially been proved also by Biggs [2] in a somewhat different setting.

The analogous theorem for vertex-induced subgraphs is the following.

3.6 Vertex Theorem. *Let* g_i, $i \geq 1$, *be a list of all isomorphism types of graphs, and let* g *and* g' *be any two graphs. Then*

$$s(g)s(g') \equiv \sum_{i\geq 1} a_i \, s(g_i),$$

where the coefficient a_i *is the number of decompositions of* $V(g_i)$ *into* $V \cup V'$ *(it need not be that* $V \cap V' = \emptyset$*) such that* $g_i[V] \simeq g$ *and* $g_i[V'] \simeq g'$. □

Several remarks are in order.

We call the identities of Theorem 3.5 *edge-identities* and those of Theorem 3.6 *vertex-identities*.

Each identity is finite, since there is no such decomposition of $E(g_i)$ or $V(g_i)$ if $\varepsilon(g_i) > \varepsilon(g)+\varepsilon(g')$ or if $\nu(g_i) > \nu(g)+\nu(g')$, respectively.

These identities have the obvious extension to products of any finite number of terms: $n(g)n(g')n(g'')\ldots n(g^{(k)})$ or $s(g)s(g')s(g'')\ldots s(g^{(k)})$.

Theorems 3.5 and 3.6 represent the combinatorial principle behind the polynomials of "All the King's Horses" [18].

4. APPLICATIONS TO RECONSTRUCTION

In this section we give direct combinatorial proofs of the results obtained via polynomials by Tutte [18].

Definition. *Any property of a graph* G *that is determined by the isomorphism types of* $G-v$, *for* $v \in V(G)$, *is said to be* reconstructible.

Any property of G *that is determined by the isomorphism types of* $G-e$, *for* $e \in E(G)$, *is said to be* edge-reconstructible.

4.1 Theorem. *The number of disconnected spanning subgraphs of* G *with the isomorphism types of the components specified is reconstructible.*

Proof. Specify the components by $g, g', g'', \ldots, g^{(k)}$. Form the edge-identity corresponding to the product $n(g)n(g')n(g'')\ldots n(g^{(k)})$. It is a linear combination of g_i, $i \geq 1$. The only such g_i with $\nu(G)$ vertices is a disconnected graph with components $g, g', g'', \ldots, g^{(k)}$.

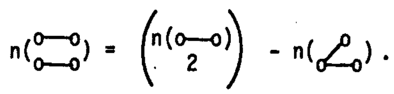
If we want each component to appear as a vertex-induced subgraph of G, we use the vertex-identity $s(g)s(g')s(g")\ldots s(g^{(k)})$. □

4.2 Corollary. *The number of one-factors of* G *is reconstructible.*

Proof. Choose each component $g, g', g", \ldots, g^{(k)}$ isomorphic to K_2. □

For example, if we rewrite Example 3.4 as follows, we find that we have reconstructed the number of one-factors in a graph on four vertices.

4.3 Example.



4.4 Theorem. *The number of hamiltonian cycles of* G *is reconstructible.*

Proof. Let $\nu(g) = n$. A hamiltonian cycle has n edges. A spanning subgraph with n edges is either

(i) disconnected, or
(ii) connected, with only one cycle.

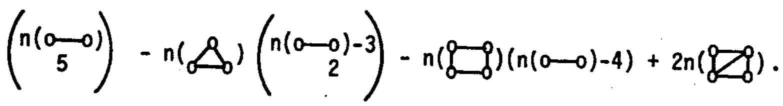
In the first case, we can count such spanning subgraphs by Theorem 4.1.

In the second case, let C_k be a cycle of length k. Form the following linear combination of edge-identities:

$$\binom{n(K_2)}{n} - \sum_{k=3}^{n-1} n(C_k)\binom{n(K_2)-k}{n-k}$$

i.e., for every cycle of lengths $3,4,\ldots,n-1$, we choose enough edges to get a graph with n edges, and then subtract these graphs from all graphs with n edges. The result is a linear combination of hamiltonian cycles, and graphs which we can count. □

4.5 Example. *The number of hamiltonian cycles in a graph with five vertices is*



Applying this formula to K_5, we find that

The number of hamiltonian cycles is

$$\binom{10}{5} - 10\binom{7}{2} - 15.6 + 2.30 = 12.$$

On the other hand, it is well-known that K_n has $(n-1)!/2$ hamiltonian cycles. When $n = 5$ this gives $4!/2 = 12$ hamiltonian cycles.

4.6 <u>Theorem</u>. *The number of separable spanning subgraphs of* G *with the isomorphism types of the blocks specified is reconstructible.*

<u>Proof</u>. Specify the blocks by $g_1, g_2, \ldots, g_k$. A separable graph g with these blocks satisfies

$$\nu(g) = 1 - k + \sum_{i=1}^{k} \nu(g_i).$$

This is essentially a statement that the block-cut-vertex tree is a tree (see Tutte [17]).

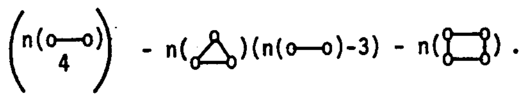
Form the edge-identity $n(g_1)n(g_2)\ldots n(g_k)$. Every spanning separable graph g with these blocks satisfies $\nu(g) = \nu(G)$ and will appear in this identity. Conversely, any connected graph g in this identity satisfying $\nu(g) = \nu(G)$ must be separable, with blocks $g_1, g_2, \ldots, g_k$, by the above formula.

If we want each block to appear as a vertex-induced subgraph of G, use the vertex-identity $s(g_1)s(g_2)\ldots s(g_k)$. □

4.7 <u>Corollary</u>. *The number of spanning trees of* G *is reconstructible.*

<u>Proof</u>. Choose each block in Theorem 4.6 to be K_2. □

4.8 <u>Example</u>. *The number of spanning trees in a graph with five vertices is:*



If we apply this to K_5, using the values from Example 4.5, we find that the number of spanning trees of K_5 is

$$\binom{10}{4} - 10.7 - 15 = 125 = 5^3,$$

which we recognize as Cayley's formula for the number of trees on 5 vertices.

<u>Definition</u>. *The* characteristic polynomial *of* G *is* $\det(\lambda I-A)$, *where* A *is the adjacency matrix of* G.

4.9 Theorem. *The characteristic polynomial of* G *is reconstructible.*

Proof. Expand $\det(\lambda I-A)$ in principal r-rowed minors of A. Each such minor is a sum over *sesquivalent* (see Tutte [18]) subgraphs of G, i.e., subgraphs of G which are disjoint unions of circuits and edges. By Theorems 4.1 and 4.4, the number of these is reconstructible.

□

4.10 Theorem. *The number of hamiltonian paths of* G *with non-adjacent ends is reconstructible.*

Proof. We use a device of Tutte [18]. Join each pair of non-adjacent vertices of G-v, for all $v \in V(G)$, by a new edge, coloured blue, say, to distinguish it from the edges of G. Use Theorem 4.4 to count the number of hamiltonian cycles of the new graph with one blue edge, and the remaining edges true edges. □

Definition. *The* rank polynomial *of* G, $R(G;z,t)$ *is*

$$R(G;z,t) = \sum_{g \subseteq G} t^{\rho(g)} z^{r(g)},$$

where the co-rank $\rho(g)$ *is the number of vertices of* g *minus the number of components of* g*, and the rank of* g *is* $r(g) = \varepsilon(g) - \rho(g)$.

4.11 Theorem. *The rank polynomial of* G *is reconstructible.*

Proof. By Tutte [18; Theorem 7.3], $R(G;t,z)$ is determined by the terms of degree less than $\rho(G)$ in t. By Lemma 3.1 and Theorem 4.1, these terms are reconstructible. □

4.12 Corollary. *The chromatic polynomial,* $P(G,u)$ *is reconstructible.*

Proof. By Tutte [18], $P(G,u) = u^{\nu(G)} R(G;-u^{-1},-1)$. □

5. RECONSTRUCTING SPANNING TREES, AND FURTHER RESEARCH

In the previous section, we used the Algebra of Subgraphs to give simple proofs of results already obtained by Tutte.

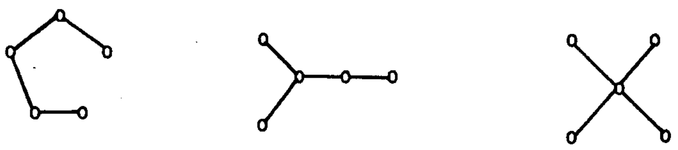
In this section, we investigate whether these results can be extended.

In Theorems 4.1 and 4.7 we reconstructed the number of spanning disconnected subgraphs of G of each isomorphism type, and the total number of spanning trees. A natural place to begin then, is with individual spanning trees.

5.1 Conjecture. Let T be a tree such that $\nu(T) = \nu(G)$. Then $n(T,G)$, the number of spanning trees of G isomorphic to T, is reconstructible.

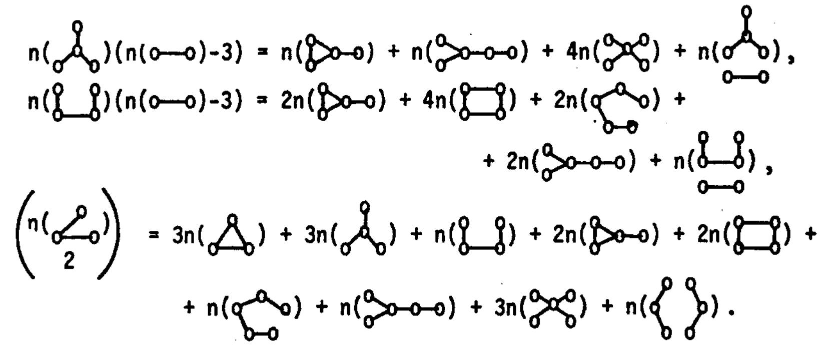
It is not known how to do this in general, but the following example from Kocay [12] is perhaps interesting.

There are three mutually non-isomorphic trees on five vertices.



T_1 T_2 T_3

We form the following three edge-identities.



The left-hand side of each identity is reconstructible. The right-hand side of each contains a linear combination of $n(T_1)$, $n(T_2)$, and $n(T_3)$, as well as a linear combination of terms which are reconstructible. Thus, we can reconstruct $n(T_1)$, $n(T_2)$, and $n(T_3)$ if we can invert the following matrix.

$$\begin{bmatrix} 0 & 1 & 4 \\ 2 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} n(T_1) \\ n(T_2) \\ n(T_3) \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} .$$

The inverse is

$$\frac{1}{6}\begin{bmatrix} -6 & -1 & 8 \\ 6 & 4 & -8 \\ 0 & -1 & 2 \end{bmatrix} .$$

In general, suppose there are m trees $T_1, T_2, \ldots, T_m$ on n vertices. With each tree we should like to associate uniquely some edge-identity which gives a linear combination of $n(T_1), n(T_2), \ldots, n(T_m)$, as well as some terms known to be reconstructible. If we can show that the resultant identities are linearly independent, then we can reconstruct $n(T_1), n(T_2), \ldots, n(T_m)$.

This is not difficult to do for trees on 6 or 7 vertices, but the author knows of no general construction.

Having reconstructed the number of spanning trees of a given type, we would then know the numbers of all subgraphs with at most $\nu(G)-1$ edges. We should then add one more edge and construct edge-identities for spanning subgraphs with $\nu(G)$ edges. We continue like this, building up one edge at a time, until we have reconstructed the edge-deleted subgraphs of G. This would prove that the edge-reconstruction and vertex-reconstruction conjectures are equivalent. One more application would then prove (we hope) that both conjectures are true.

Clearly if this approach is to work, there must be enough linearly independent edge-identities.

5.2 Theorem. *The number of linearly independent edge-identities available for reconstructing graphs with* n *vertices and* e *edges is at most the number of disconnected graphs with no isolated vertices,* e *edges, and at least* n *vertices, such that each component has at most* n *vertices.*

Proof. We wish to use an edge-identity of the form $n(g_1)n(g_2)\ldots n(g_k)$ for reconstructing graphs with n vertices and e edges. Clearly we must have $\nu(g_i) \le n$ for each $i = 1,2,\ldots,k$. We must also have

$$\nu(g_1) + \nu(g_2) + \ldots + \nu(g_k) \ge n,$$

or we do not obtain spanning subgraphs. Similarly we must have

$$\varepsilon(g_1) + \varepsilon(g_2) + \ldots + \varepsilon(g_k) = e,$$

since we want each identity to contain a linear combination of graphs with e edges, but no graphs with more than e edges.

If some g_i is disconnected, with components $g, g', g'', \ldots, g^{(\ell)}$, say, then we can replace $n(g_i)$ by $n(g)n(g')n(g'')\ldots n(g^{(\ell)})$ and not change the number of independent identities.

We now see that each identity corresponds uniquely to a disconnected graph with components $g_1, g_2, \ldots, g_k$ which satisfies the conditions of the theorem. Similarly every such disconnected graph corresponds to a unique edge-identity. □

We can now use Theorem 5.2 to count the number of edge-identities, and compare this number with the number of spanning subgraphs (i.e.,

ε-ν \ ν	2	3	4	5	6	7	8	9	10	11
-5									1	1
-4							1	1	3	7
-3					1	1	3	7	17	39
-2			1	1	3	6	15	33	83	202
-1	1	1	2	4	9	20	50	124	332	895
0		1	2	5	15	41	124	369	1132	3491
1			1	5	20	73	271	974	3507	12487
2			1	4	22	110	515	2272	9777	40752
3				2	20	133	832	4683	24543	121470
4				1	14	139	1181	8563	55703	331374
5				1	9	126	1460	13969	114550	828313
6					5	95	1581	20376	213590	1900732
7					2	64	1516	26675	362763	4012988
8					1	40	1291	31423	562151	7811406
9					1	21	970	33377	796325	14046888

Table 5.1

v(ε,ν), the number of graphs with no isolated vertices.

ε-ν \ ν	2	3	4	5	6	7	8	9	10	11
-5									1	2
-4							1	2	5	12
-3					1	2	5	12	29	69
-2			1	2	5	11	27	62	152	373
-1	0	1	2	6	14	36	89	229	599	1609
0		2	6	14	38	97	264	728	2084	6100
1			11	32	87	247	716	2155	6694	21461
2			20	62	192	584	1850	6022	20414	71358
3				117	396	1322	4528	16080	59331	226654
4				204	795	2873	10706	41135	165362	689930
5				351	1543	6163	24649	102057	444125	2019741
6					2938	12948	56087	247947	1158753	5709066
7					5466	26873	126624	596609	2963600	15676412
8					9982	54851	284228	1431433	7505618	42133036
9					18087	110380	632129	3435582	18979618	111838422

Table 5.2

e(ν,ε), the number of edge-identities (see Theorem 5.2).

ε−ν \ ν	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
-9																	1	2
-8															1	2	1.67	1.71
-7													1	2	1.67	1.71	1.67	1.71
-6											1	2	1.67	1.71	1.67	1.71	1.66	1.67
-5									1	2	1.67	1.71	1.67	1.71	1.67	1.68	1.64	1.63
-4							1	2	1.67	1.71	1.67	1.73	1.68	1.69	1.66	1.65	1.62	1.60
-3					1	2	1.67	1.71	1.71	1.77	1.71	1.73	1.69	1.68	1.66	1.64	1.61	1.59
-2			1	2	1.67	1.83	1.80	1.88	1.83	1.85	1.80	1.78	1.74	1.71	1.68	1.65	1.62	1.60
-1	0	1	1	1.5	1.56	1.80	1.78	1.85	1.80	1.80	1.76	1.73	1.69	1.67	1.64	1.61	1.59	1.57
0		2	3	2.8	2.53	2.37	2.13	1.97	1.84	1.75	1.67	1.62	1.58	1.55	1.53	1.51	1.50	1.49
1			11	6.4	4.35	3.38	2.64	2.21	1.91	1.72	1.59	1.50	1.45	1.41	1.39	1.38	1.37	1.37
2			20	15.5	8.73	5.31	3.59	2.65	2.09	1.75	1.54	1.41	1.33	1.28	1.26	1.25	1.24	1.25
3				58.5	19.8	9.94	5.44	3.43	2.42	1.87	1.55	1.36	1.25	1.18	1.14	1.13	1.12	1.13
4				204	56.8	20.7	9.07	4.80	2.97	2.08	1.61	1.35	1.20	1.11	1.05	1.03	1.02	1.02
5				351	171	48.9	16.9	7.31	3.88	2.44	1.75	1.38	1.18	1.06	0.99	0.95	0.94	0.93
6					587	136	35.5	12.2	5.43	3.00	1.97	1.46	1.19	1.04	0.95	0.89	0.87	0.86
7					2723	420	83.5	22.4	8.17	3.91	2.31	1.60	1.24	1.04	0.92	0.86	0.82	0.80
8					9982	1371	220	45.6	13.4	5.39	2.83	1.80	1.32	1.06	0.92	0.83	0.78	0.76
9					18087	5256	652	103	28.8	7.96	3.63	2.10	1.44	1.11	0.93	0.82	0.76	0.72
10														1.19	0.96	0.83	0.75	0.70
11															1.01	0.84	0.75	0.69
12																0.87	0.75	0.68
13																	0.77	0.69
14																		0.69

Table 5.3

$e(\nu,\varepsilon)/v(\nu,\varepsilon)$.

graphs with no isolated vertices) with e edges.

The following three tables are taken from Kocay [12]. Table 5.1 counts the number of graphs with ν vertices and ε edges and no isolated vertices. Table 5.2 gives the corresponding number of edge-identities, and Table 5.3 gives the ratio of the two.

The interesting fact is that the ratio can be less than one. If the ratio becomes small enough, does this mean that the reconstruction conjectures are false?

We finish this section with a question related to enumeration. It is well-known (see Doubilet, Rota, and Stanley [5]) that corresponding to different enumerative systems there are different classes of generating functions. Combinatorial operations on the sets of objects being counted correspond to algebraic operations on the generating functions. For example (see [5]) problems involving the natural numbers with the usual order relation give rise to ordinary generating functions. Problems involving subsets of a set correspond to exponential generating functions, and divisibility problems correspond to Dirichlet generating functions. In terms of generating functions, inclusion-exclusion becomes particularly simple.

Since we are really using inclusion-exclusion in Theorem 4.4, Corollary 4.7, and Examples 4.5 and 4.8, the following question seems natural.

5.3 Problem. *Is there a class of generating functions for which subgraph counting becomes algebraic?*

II. THE METHOD OF PARTIAL AUTOMORPHISMS

6. PARTIAL AUTOMORPHISMS AND PSEUDO-SIMILAR VERTICES

Let G and H be graphs with some "large" isomorphic subgraphs $g \subseteq G$ and $h \subseteq H$, where $g \cong h$. For example, G and H might be reconstructions of each other, where g and h are isomorphic vertex-deleted subgraphs: $G-u \cong H-v$; or G and H might be edge-reconstructions of each other, for which g and h are isomorphic edge-deleted subgraphs: $G-e \cong H-f$.

Let $p : g \to h$ be an isomorphism. We want to embed G and H into some larger graph Γ such that the isomorphism p can be extended to an automorphism of Γ. We make the following definition.

Definition. *Let* Γ *be a graph, and let* p *be an isomorphism between two subgraphs* g *and* h *of* Γ. p *is a* partial automorphism *of* Γ *if the isomorphism* $p : g \to h$ *can be extended to an automorphism of* Γ.

The symmetry of Γ will then force some of the structure of G and H in the above situation.

This technique was first used in connection with pseudo-similar vertices in a graph (see Godsil and Kocay [7]).

Definition. *Let* $u,v \in V(G)$. *If there is an automorphism* $p \in \text{Aut } G$ *such that* $p(u) = v$, *then* u *and* v *are* similar.

If u and v are similar vertices of G, say $p(u) = v$, then $G\text{-}u \cong G\text{-}v$; for $p(G\text{-}u) = G\text{-}v$. One purported "proof" of the reconstruction conjecture rested on the assumption that if $G\text{-}u \cong G\text{-}v$, then there exists $p \in \text{Aut } G$ such that $p(u) = v$ (see Harary and Palmer [8]).

The falsity of this assumption is demonstrated by the following famous graph, in which $G\text{-}u \cong G\text{-}v$ but u and v are not similar.

G:

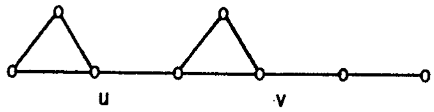


Figure 6.1

Definition. *Let* $u,v \in V(G)$. *If* $G\text{-}u \cong G\text{-}v$ *but there exists no* $p \in \text{Aut } G$ *such that* $p(u) = v$, *then* u *and* v *are* pseudo-similar *vertices.*

Why then, do pseudo-similar vertices exist? We can answer this question with partial automorphisms.

6.1 Lemma. *Let* u *and* v *be pseudo-similar vertices in a graph* G, *and let* $p : G\text{-}v \to G\text{-}u$ *be an isomorphism. Then there exists a positive integer* k *such that* $p^k(u) = v$.

Proof. First note that p maps $V(G)\text{-}\{v\}$ onto $V(G)\text{-}\{u\}$. Consider $p(u)$. Either $p(u) = v$, or we can find $p^2(u)$. Either $p^2(u) = v$, or we can find $p^3(u)$. We must eventually have $p^k(u) = v$, for some positive k, since p is one-to-one and onto, and $V(G)$ is a finite set. □

Note that in terms of the graphs G, H, g, and h above, we have $G = H$, $g = G\text{-}v$ and $h = G\text{-}u$.

We denote the set $\{u,p(u),p^2(u),\ldots,p^k(u) = v\}$ by $[u,v]$. Notice that it has a natural order specified by the mapping p.

6.2 Theorem. *Let* u *and* v *be pseudo-similar vertices in* G. *Let* $p : G-v \to G-u$ *be an isomorphism. Then* G *can be embedded as an induced subgraph (i.e., vertex-induced subgraph) into a graph* Γ *for which:*

(i) u *and* v *are similar;*

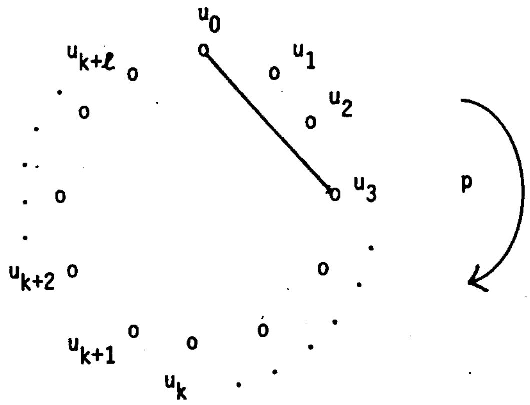
(ii) p *is a partial automorphism of* Γ.

Before proving Theorem 6.2, we notice that part (ii) actually subsumes part (i); for once p is extended to an automorphism of Γ, it will still satisfy $p^k(u) = v$, by Lemma 6.1, thereby showing that u and v are similar in Γ.

Proof of Theorem 6.2. Let $V' = V(G) - [u,v]$. Write $u_i = p^i(u)$ for $i = 0,1,2,\ldots,k$, so that $[u,v] = \{u_0,u_1,u_2,\ldots,u_k\}$. Note that u and v are similar in $G[u,v]$ (the graph induced by $[u,v]$), since the mapping $\psi : [u,v] \to [u,v]$ defined by $\psi(u_i) = u_{k-i}$, $i = 0,1,2,\ldots,k$, is an automorphism of $G[u,v]$. But u and v are pseudo-similar in G, so $G \neq G[u,v]$. Therefore $V' \neq \emptyset$.

Now p is an automorphism of $G[V']$ and so has finite period r on $G[V']$. Add a sequence of ℓ new vertices $u_{k+1},u_{k+2},\ldots,u_{k+\ell}$, until $k+\ell+1$ is a multiple of r, and $\ell \geq k$.

Extend p by defining $p(u_k) = u_{k+1}$, $p(u_{k+1}) = u_{k+2}$, $\ldots$, $p(u_{k+\ell}) = u_0$. Attach edges to $u_{k+1},u_{k+2},\ldots,u_{k+\ell}$ appropriately, so that p acts as an automorphism of the new graph Γ so obtained. This adds no new edges amongst V(G), since $\ell \geq k$. For let $(u_0,u_i) \in E(G)$. Geometrically, this represents a chord of a circle.



The effect of p is to translate this chord in a clockwise direction. Since $\ell \geq k$, this cannot create any new edges in $G[u,v]$ or $G[V']$.

The extension of p acts as an automorphism of Γ under which u and v are similar. This completes the proof of the theorem. □

It is now easy to see why u and v are pseudo-similar in G. They are similar in Γ, and to get G we have deleted a set $A = \{u_{k+1}, u_{k+2}, \ldots, u_{k+\ell}\}$ of consecutive vertices of an orbit of p such that $p(A \cup \{v\}) = A \cup \{u\}$.

We remark that the graph Γ is not unique; for we can increase ℓ by any multiple of r and still get a graph satisfying these conditions. Also it is sometimes not necessary to take $\ell \geq k$.

It is a simple exercise to apply this technique to the graph of Figure 6.1 to get the following graph Γ.

Γ:

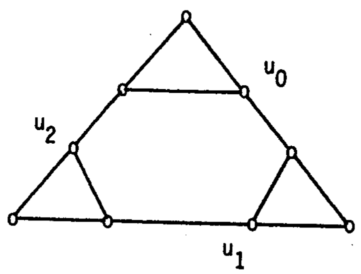


Figure 6.2

A catalogue of graphs with pseudo-similar vertices can be found in Kocay [13].

6.3 <u>Theorem</u>. *Let* u, v, p, *and* G *be as in Theorem 6.2. Let* $V' = V(G) - [u,v]$. *Then* $|V'| \geq 6$.

<u>Proof</u>. We know that $V' \neq \emptyset$. There are thus five main cases to be considered.

Let $\psi : [u,v] \to [u,v]$ be the automorphism of $G[u,v]$ defined in Theorem 6.2.

<u>Case I</u>. $|V'| = 1$. Let $V' = \{a\}$. Then every $u_i \in [u,v]$ can be taken adjacent to a. But then $(a)\psi$, the product of ψ and the permutation (a) is an automorphism of G taking u to v, a contradiction.

<u>Case II</u>. $|V'| = 2$. Let $V' = \{a,b\}$. Then p acts as (ab) on V', or we are reduced to Case I. We can assume that u_o is joined to one of a and b, but not both, for otherwise $(a)(b)\psi \in \text{Aut } G$. Without loss of generality, take u_o adjacent to a, but not to b. Then u_1 is adjacent to b, but not to a, etc.

If k is even, then u_k is adjacent to a but not to b. In this case $(a)(b)\psi \in \text{Aut } G$.

If k is odd, then u_k is adjacent to b but not to a. Then $(ab)\psi \in \text{Aut } G$.

<u>Case III</u>. $|V'| = 3$. Let $V' = \{a,b,c\}$. Then p acts as (abc) on V', for otherwise it would have a fixed point. $G[V']$ is either a triangle or three independent vertices. In either case, $\text{Aut } G[V'] \cong S_3$, the symmetric group on V'. u_0 is joined either to one or two of a, b, and c. By taking complements if necessary, we can take u_0 adjacent to a only. Then u_1 is adjacent to b, and u_2 is adjacent to c, etc.

If $k \equiv 0 \pmod 3$, then $(a)(bc)\psi \in \text{Aut } G$.

If $k \equiv 1 \pmod 3$, then $(ab)(c)\psi \in \text{Aut } G$.

If $k \equiv 2 \pmod 3$, then $(ac)(b)\psi \in \text{Aut } G$.

<u>Case IV</u>. $|V'| = 4$. Let $V' = \{a,b,c,d\}$. In this case p must act either as $(abcd)$, or as $(ab)(cd)$ on V'. By taking complements if necessary we can suppose that u_0 is joined to one or two of a, b, c, and d.

(i) Suppose p acts as $(ab)(cd)$ on $G[V']$. If u_0 is joined only to one of a, b, c, and d, say a, then this case reduces to Case II. If u_0 is joined to two of a, b, c, and d, then we can assume it is a and c, or a and d. Either possibility reduces to Case II.

(ii) Suppose p acts as $(abcd)$ on $G[V']$. Then $G[V']$ is one of the following graphs, or their complements.

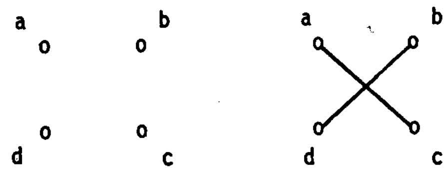


Figure 6.3

In either case, $\text{Aut } G[V'] \geq D_4$, the dihedral group on V'.

Without loss of generality, we can assume that one of the following three choices holds: u_0 is joined only to a; u_0 is joined only to a and b; u_0 is joined only to a and c. In any case, we have the following.

If $k \equiv 0 \pmod 4$, then $(a)(c)(bd)\psi \in \text{Aut } G$.

If $k \equiv 1 \pmod 4$, then $(ab)(cd)\psi \in \text{Aut } G$.

If $k \equiv 2 \pmod 4$, then $(ac)(b)(d)\psi \in \text{Aut } G$.

If $k \equiv 3 \pmod 4$, then $(ac)(bd)\psi \in \text{Aut } G$.

Thus in every possibility, u and v are similar in G, not pseudo-similar, a contradiction.

Case V. $|V'| = 5$. Let $V' = \{a,b,c,d,e\}$. In this case p must act either as (abcde) or as (abc)(de) on V'. We can assume that u_o is joined to one or two of a, b, c, d, and e.

(i) Suppose that p acts as (abc)(de) on V'. It is then easy to see that in G[V'], either there are no edges joining {a,b,c} to {d,e} or all possible edges are present. Thus Aut G[V'] factors into (Aut G[{a,b,c}]) × (Aut G[{d,e}]). This then reduces to Cases II and III.

(ii) Suppose p acts as (abcde) on V'. Then G[V'] is one of: an empty graph; a complete graph; or a pentagon. In each case Aut $G[V'] \geq D_5$, the dihedral group acting on V'.

If $k \equiv 0 \pmod 5$, then $(a)(be)(cd)\psi \in \text{Aut } G$.

If $k \equiv 1 \pmod 5$, then $(ae)(bd)(c)\psi \in \text{Aut } G$.

If $k \equiv 2 \pmod 5$, then $(ac)(b)(de)\psi \in \text{Aut } G$.

If $k \equiv 3 \pmod 5$, then $(ad)(bc)(e)\psi \in \text{Aut } G$.

If $k \equiv 4 \pmod 5$, then $(ae)(bd)(c)\psi \in \text{Aut } G$.

Thus in each case, u and v turn out to be similar, instead of pseudo-similar. This completes the proof of the theorem. □

Note that $|V'| = 6$ is possible. This is the case of the graph G of Figure 6.1.

Several properties of the degree sequences of graphs with pseudo-similar vertices are discussed in Kocay [14].

7. PSEUDO-SIMILAR EDGES

A result analogous to Theorem 6.2 holds for pseudo-similar edges.

Definition. *Let* $e,f \in E(G)$. *If there exists* $p \in \text{Aut } G$ *such that* $p(e) = f$, *then* e *and* f *are* similar *edges of* G.

Definition. *Let* $e,f \in E(G)$. *If* $G-e \cong G-f$ *but* e *and* f *are not similar in* G, *then* e *and* f *are* pseudo-similar *edges.*

In the graph G of Figure 7.1 edges e and f are pseudo-similar. In fact, so are vertices u and v.

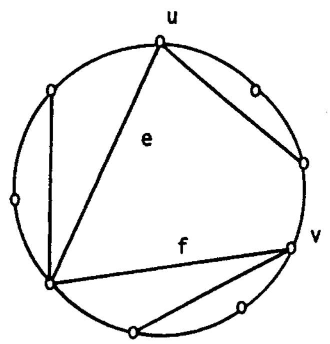


Figure 7.1

7.1 Theorem. *Let* e *and* f *be pseudo-similar edges in a graph* G. *Let* p : G-f → G-e *be an isomorphism. Then* G *can be embedded as an edge-induced subgraph into a graph* Γ *such that:*

(i) e *and* f *are similar in* Γ*;*

(ii) p *is a partial automorphism of* Γ*;*

(iii) the edges of Γ\G *are contained in the orbit of* e *and* f *under* <p>*, the group generated by* p.

Proof. Let e = uv ∈ E(G) and f = ab ∈ E(G). Note that u, v, a, and b need not necessarily be four distinct vertices, and that p is a permutation of V(G).

Since p is an isomorphism, we can show, as in Lemma 6.1, that

$$\{a,b\} = p^{k_1}\{u,v\} \quad \text{and} \quad \{u,v\} = p^{k_2}\{a,b\}$$

for positive integers k_1 and k_2, since p must take edges to edges and non-edges to non-edges.

Note that $\{a,b\},p\{a,b\},p^2\{a,b\},\ldots,p^{k_2-1}\{a,b\}$ is a sequence of non-edges in G-f. If we add edges joining these pairs of vertices in the graph G to get Γ, then p becomes an automorphism of Γ, for which $p^{k_1}(e) = f$ and $p^{k_2}(f) = e$, thereby proving the theorem. □

Note that unlike pseudo-similar vertices, we can look at pseudo-similar edges from two points of view. Instead of adding the edges $ab, p(ab), p^2(ab),\ldots,$ to G to get Γ, we could have removed the edges $uv, p(uv), p^2(uv),\ldots,$ to get Γ', a subgraph of G, on which p acts as an isomorphism.

8. RECONSTRUCTING BI-DEGREED GRAPHS

If G is a k-regular graph, then for any $v \in V(G)$, G-v has exactly k vertices of degree k-1, i.e., there is only one way to rejoin v to get a regular graph. Thus, regular graphs are reconstructible.

If G has only two degrees, k and ℓ, where $k \geq \ell$, then the deletion of a vertex v of degree k will leave a total of k vertices of degrees k-1 or ℓ-1. If $\ell \neq k-1$, there is only one way to rejoin v. Thus it is natural to look at graphs with only two degrees, k and k-1. We call such a graph a *bi-degreed* graph.

If k = 2, then G is a collection of paths, which is not very interesting. Therefore we take k = 3. We look at several simple examples.

If there is only one vertex v of degree two, and the rest have degree three, then G-v has exactly two vertices of degree two, and G is reconstructible.

Similarly, if there are two adjacent vertices, u and v, of degree two then G-u has a vertex of degree one, and one of degree two. Again G is reconstructible (see Figure 8.1), since it is well-known that the degrees of the neighbours of a vertex are reconstructible.

G:

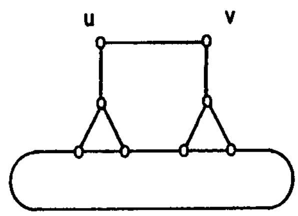


Figure 8.1

If u and v are not adjacent in G, then it is not in general known whether G is reconstructible. However, much of the structure of G can be determined in this case, for general k. This can be found in Kocay [15].

We look now at the case when G has three vertices of degree two: u, v, and z. Clearly we can assume that no one of these is joined to both of the other two.

Consider the case when u is joined to z, but not to v. This is illustrated in Figure 8.2.

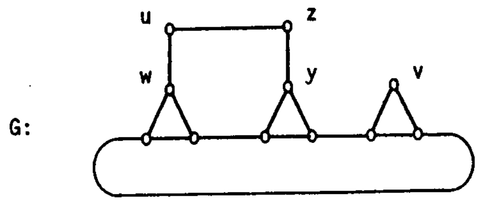


Figure 8.2

If G is not reconstructible from G-u, then it must be that a new vertex x is attached to G-u at z and v to get H, a reconstruction of G.

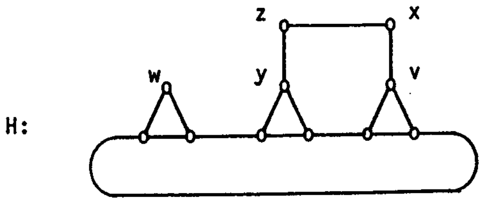
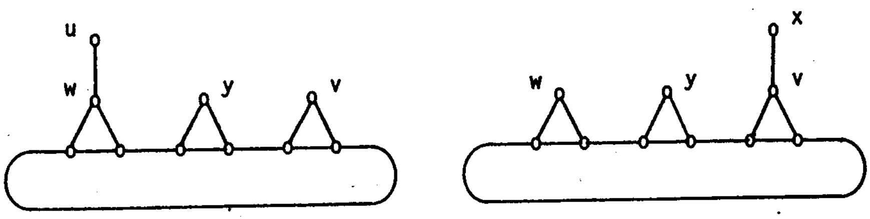


Figure 8.3

We now have G-u = H-x. Consequently G-z ≅ H-z if G and H are to be reconstructions of each other.



G-z H-z

Figure 8.4

We can now use the method of partial automorphisms. Let $p : G\text{-}z \to H\text{-}z$ be an isomorphism.

8.1 Lemma. $p(u) = x$,
$p(w) = v$,
$p\{y,v\} = p\{w,y\}$. □

8.2 Theorem. *Let* G *and* H *be as above. Then* G ≅ H.

Proof. By Lemma 8.1 there are two cases to consider.

Case I. p(y) = y and p(v) = w.

Consider $\Gamma = G \cup H$.

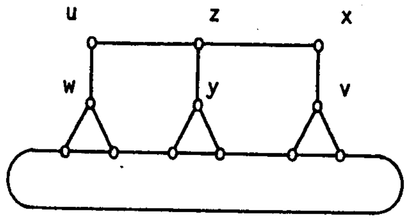


Figure 8.5

We have $p : V(\Gamma) - \{x,z\} \to V(\Gamma) - \{u,z\}$. Extend p to $V(\Gamma)$ by setting $p(z) = z$ and $p(x) = u$. Then $p \in \text{Aut}\,\Gamma$, by Lemma 8.1. But p acts on $\{u,v,w,x,y,z\}$ as $(y)(z)(wv)(ux)$. Therefore

$$G = \Gamma - x = p(\Gamma - u) = p(H),$$

so that $G \cong H$.

Case II. p(y) = w and p(v) = y.

Consider $\Gamma = G \cup H + ux$.

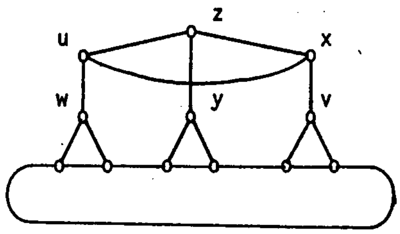


Figure 8.6

As in Case I, $p : V(\Gamma) - \{x,z\} \to V(\Gamma) - \{u,z\}$. Extend p to $V(\Gamma)$ by setting $p(x) = z$ and $p(z) = u$. Then $p \in \text{Aut}\,\Gamma$. But p acts on $\{u,v,w,x,y,z\}$ as $(uxz)(wvy)$. Therefore

$$G = \Gamma - x = p(\Gamma - u) = p(H),$$

or $G \cong H$. □

We see that the isomorphism p extends to an automorphism of a graph Γ which contains both G and H as subgraphs. The automorphism of Γ guarantees that $G \cong H$.

Note that we have only used the following information to reconstruct G: the degree sequence of G; the vertex-deleted subgraphs

G-u (= H-x), G-z, and H-z; and the isomorphism $p : G-z \to H-z$.

In a more general situation, it is reasonable to assume that we should need to consider several isomorphisms $p_1, p_2, \ldots, p_m$ between several pairs of vertex-deleted subgraphs. The group of symmetries generated by $p_1, p_2, \ldots, p_m$ should then force most of the structure of G and H.

REFERENCES

[1] N. Biggs, On the algebra of graph types, in *Graph Theory and Related Topics*, Ed. J.A. Bondy and U.S.R. Murty, Academic Press, New York, 1979 81-89.

[2] N. Biggs, On cluster expansions in graph theory and physics, *Quart. J. Math. Oxford* (2), 29 (1978) 159-173.

[3] J.A. Bondy and R.L. Hemminger, Graph reconstruction - a survey, *J. of Graph Theory*, 1 (1977) 227-268.

[4] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Macmillan Press, Ltd., London, 1976.

[5] P. Doubilet, G.-C. Rota, and R. Stanley, On the foundations of combinatorial theory (VI): The idea of generating function, *Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol.II: Probability Theory*, Univ. of California, 1972 267-318.

[6] P. Erdös, L. Lovász, and J. Spencer, Strong independence of graphcopy functions, in *Graph Theory and Related Topics*, Editors J.A. Bondy and U.S.R. Murty, Academic Press, New York, 1979 165-172.

[7] C. Godsil and W.L. Kocay, Constructing graphs with pairs of pseudo-similar vertices, submitted to *J. Combin. Theory (B)*.

[8] F. Harary and E. Palmer, On similar points of a graph, *J. Math. Mechanics*, 15 (1966) 623-630.

[9] P.J. Kelly, A congruence theorem for trees, *Pacific J. Math.* 7 (1957) 961-968.

[10] W.L. Kocay, 2-reconstruction of trees, *Ars Combinatoria*, 7 (1979) 47-134.

[11] W.L. Kocay, An extension of Kelly's Lemma to spanning subgraphs, *Cong. Num.* 31 (1981) 109-120.

[12] W.L. Kocay, On reconstructing spanning subgraphs, *Ars Combinatoria,* 11 (1981) 301-313.

[13] W.L. Kocay, On pseudo-similar vertices, *Ars Combinatoria* 10 (1980) 147-163.

[14] W.L. Kocay, On reconstructing degree sequences, *Utilitas Math.* 17 (1980) 151-162.

[15] W.L. Kocay, Partial automorphisms and the reconstruction conjecture, in preparation.

[16] C.St.J.A. Nash-Williams, The reconstruction problem, in *Selected Topics in Graph Theory,* Eds. L.W. Beineke and R.J. Wilson, Academic Press, 1978 205-236.

[17] W.T. Tutte, *Connectivity in Graphs*, Univ. of Toronto Press, Toronto, 1966.

[18] W.T. Tutte, All the king's horses, in *Graph Theory and Related Topics,* Eds. J.A. Bondy and U.S.R. Murty, Academic Press, New York, 1979 15-33.

[19] H. Whitney, The coloring of graphs, *Ann. of Mathematics* 33 (1932) 668-718.

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