Butterfly Factorizations of K_8

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Abstract

We determine the number of non-isomorphic butterfly factorizations of K_8.

1 Introduction

Butterfly factorizations of K_{2n} were introduced in [1]. For K_{2n}, one has n butterflies, each containing a body and two wings. The body is one pair, and each wing contains $n-1$ pairs. So the entire butterfly uses $2n-1$ pairs. Moreover the n bodies are an ordinary one-factor of K_{2n} and each wing, with its associated body, is also a one-factor. The following example shows a butterfly factorization of K_8.

37 46 58	**12**	35 47 68
18 26 57	**34**	15 28 67
17 24 38	**56**	13 27 48
16 23 45	**78**	14 25 36

The central pair (in bold) of each row forms the body of each butterfly. Each body with its right or left wing forms a 1-factor, and the bodies also form a 1-factor.

In general, let AA BB CC DD be the bodies of a butterfly factorization of K_8. AA represents A_1A_2, and similarly for the other pairs. Then the butterfly factorization can be represented as:

$$\begin{array}{ccccccc} BC & BD & CD & AA & BC & BD & CD \\ AC & AD & CD & BB & AC & AD & CD \\ AB & AD & BD & CC & AB & AD & BD \\ AB & AC & BC & DD & AB & AC & BC \end{array}$$

where each occurrence of B represents one of the points in the B-body, and so forth.

We note that similar pairs (eg., pairs BC) in a butterfly may involve either 3 or 4 symbols. Thus one might have pairs like B_1C_1 and B_1C_2 that involve 3 symbols or pairs like B_1C_1 and B_2C_2 that involve 4 symbols.

Lemma 1. *Any butterfly must have at least one set of similar pairs that involve 4 symbols.*

Proof: Suppose it is possible to have only 3 symbols in all similar pairs. Then we could write the first butterfly as

$$B_1C_1 \;\; B_2D \;\; C_2D \;\; AA \;\; B_1C_2 \;\; B_2D \;\; C_1D.$$

There is no loss in generality in assigning D so that we have B_2D_1 and B_2D_2. This forces the remaining pairs to be C_2D_2 and C_1D_1, thus creating a set of pairs involving 4 symbols. So it is not possible to have only 3 symbols in all similar pairs.

2 Discussion of K_8

So, to build up a butterfly factorization of K_8, we may start with

$$B_1C_1 \;\; B_2D \;\; C_2D \;\; AA \;\; B_2C_2 \;\; B_1D \;\; C_1D$$

There is no loss of generality in assigning B_2D_1 C_2D_2 to give

$$B_1C_1 \;\; B_2D_1 \;\; C_2D_2 \;\; AA \;\; B_2C_2 \;\; B_1D \;\; C_1D$$

Now there are two possibilities.

Case 1.

$$B_1C_1 \;\; B_2D_1 \;\; C_2D_2 \;\; AA \;\; B_2C_2 \;\; B_1D_1 \;\; C_1D_2$$

Case 2.

$$B_1C_1 \;\; B_2D_1 \;\; C_2D_2 \;\; AA \;\; B_2C_2 \;\; B_1D_2 \;\; C_1D_1$$

In Case 1, we proceed as follows. The CD pairs in the first butterfly force the CD pairs in the second butterfly (subject to interchanging the right and left wings); BD pairs in the first butterfly force the BD pairs in the third butterfly; the BC pairs in the first butterfly force the BC pairs in the fourth butterfly. Then assigning one of the occurrences of A in each butterfly forces the remaining occurrences of A, to give the following array.

$$\begin{array}{lllllll}
B_1C_1 & B_2D_1 & C_2D_2 & AA & B_2C_2 & B_1D_1 & C_1D_2 \\
A_1C_2 & A_2D_2 & C_1D_1 & BB & A_2C_1 & A_1D_2 & C_2D_1 \\
A_1B_2 & A_2D_1 & B_1D_2 & CC & A_2B_1 & A_1D_1 & B_2D_2 \\
A_2B_2 & A_1C_1 & B_1C_2 & DD & A_1B_1 & A_2C_2 & B_2C_1
\end{array}$$

So there is a single solution in Case 1. Call it Butterfly 1.

In Case 2, we have the first butterfly, and can now assign the CD pairs in the second, the BD pairs in the third, and the BC pairs in the fourth.

$$\begin{array}{lllllll}
B_1C_1 & B_2D_1 & C_2D_2 & AA & B_2C_2 & B_1D_2 & C_1D_1 \\
AC_2 & AD_1 & C_1D_2 & BB & AC_1 & AD_2 & C_2D_1 \\
AB_2 & AD_2 & B_1D_1 & CC & AB_1 & AD_1 & B_2D_2 \\
AB_2 & AC_1 & B_1C_2 & DD & AB_1 & AC_2 & B_2C_1
\end{array}$$

Now there is no loss of generality in assigning A_1 and A_2 in the left wing of the second butterfly and this gives the right wing of the third and fourth. We are now at the following stage.

$$\begin{array}{lllllll}
B_1C_1 & B_2D_1 & C_2D_2 & AA & B_2C_2 & B_1D_2 & C_1D_1 \\
A_1C_2 & A_2D_1 & C_1D_2 & BB & AC_1 & AD_2 & C_2D_1 \\
AB_2 & AD_2 & B_1D_1 & CC & A_2B_1 & A_1D_1 & B_2D_2 \\
AB_2 & AC_1 & B_1C_2 & DD & A_1B_1 & A_2C_2 & B_2C_1
\end{array}$$

Note that there remain two pairs AC_1, two pairs AB_2, two pairs AD_2. Assign either A_1 or A_2 to one of the AC_1 pairs, and all the other pairs are determined. Thus we have obtained two solutions in Case 2. If we assign A_1C_1 to the B-butterfly, call the resulting factorization Butterfly 2:

$$\begin{array}{ccccccc} B_1C_1 & B_2D_1 & C_2D_2 & AA & B_2C_2 & B_1D_2 & C_1D_1 \\ A_1C_2 & A_2D_1 & C_1D_2 & BB & A_1C_1 & A_2D_2 & C_2D_1 \\ A_2B_2 & A_1D_2 & B_1D_1 & CC & A_2B_1 & A_1D_1 & B_2D_2 \\ A_1B_2 & A_2C_1 & B_1C_2 & DD & A_1B_1 & A_2C_2 & B_2C_1 \end{array}$$

If we assign A_2C_1 to the B-butterfly, call the result Butterfly 3:

$$\begin{array}{ccccccc} B_1C_1 & B_2D_1 & C_2D_2 & AA & B_2C_2 & B_1D_2 & C_1D_1 \\ A_1C_2 & A_2D_1 & C_1D_2 & BB & A_2C_1 & A_1D_2 & C_2D_1 \\ A_1B_2 & A_2D_2 & B_1D_1 & CC & A_2B_1 & A_1D_1 & B_2D_2 \\ A_2B_2 & A_1C_1 & B_1C_2 & DD & A_1B_1 & A_2C_2 & B_2C_1 \end{array}$$

In order to determine the isomorphism classes of these butterfly factorizations, we have represented them as graphs. Map the points $(A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2)$ to vertices $(1, 2, 3, 4, 5, 6, 7, 8)$, in that order. Construct a graph K_8 on this vertex set, and remove the 1-factor $\{15, 26, 37, 49\}$, which represents $\{AA, BB, CC, DD\}$. Subdivide the remaining edges, so that each edge can be represented by a vertex at its midpoint. Now add 4 more vertices a, b, c, d to represent the A, B, C and D butterflies. Vertex a is joined to the subdividing vertex (midpoint) of the pairs in the wings of the A-butterfly, and so forth. Any automorphism of the graph that maps $\{a, b, c, d\}$ to itself is an automorphism of the butterfly factorization, and conversely. Thus we need only find the automorphism groups of these graphs to compare the factorizations for isomorphism. We have used the Groups & Graphs [2] software for this purpose.

Lemma 2. *Butterflies 1 and 2 are isomorphic, with automorphism groups of order 12. Butterfly 3 has an automorphism group of order 48.*

A mapping from the points of Butterfly 2 to Butterfly 1 is given by $\binom{A_1A_2B_1B_2C_1C_2D_1D_2}{D_1D_2B_2B_1C_1C_2A_1A_2}$, where the top row gives the point of Butterfly 2, and the bottom row gives the corresponding points of Butterfly 1.

We can therefore state the lemma:

Lemma 2. *There are two non-isomorphic butterfly factorizations of* K_8.

References

[1] W.L. Kocay and R.G. Stanton, On Butterfly Factorizations, *Bulletin of the Institute of Combinatorial Mathematics and its Applications* **55** (2009), 57–65.

[2] W.L. Kocay, Groups & Graphs – Software for Graphs, Digraphs, and their Automorphism Groups, *Match* **58 #2** (2007), 431-443.