Degree sequence conditions for partial Steiner triple systems

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A *partial Steiner triple system* (PSTS) of order n is a collection of 3-element subsets of the vertex set $\{1, 2, \ldots, n\}$ called *triples* that pairwise intersect in at most one vertex. If $\mathcal{H}$ is a PSTS and x is a vertex, then the degree of x is d_x and is the number of triples in $\mathcal{H}$ that contain x. The sequence $D = (d_1, d_2, \ldots, d_n)$ is called the degree sequence of the PSTS $\mathcal{H}$, and we assume without loss that $d_1 \geq d_2 \geq \ldots \geq d_n$.

Theorem *Let $D = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a* PSTS $\mathcal{H}$, *where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $\sum_i^n d_i \equiv 0 \pmod 3$, and the following conditions hold for $k = 1, 2, \ldots, n$.*

$$\sum_{i=1}^{k} d_i \leq \frac{3}{2}\binom{k}{2} + \frac{1}{2}\sum_{j=k+1}^{n} \min\{k, d_j\}, \qquad \text{if } k \leq \frac{n}{2}$$

$$\sum_{i=1}^{k} d_i \leq \binom{k}{2} + \frac{1}{2}(n-k)\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2}\sum_{j=k+1}^{n} \min\{k, d_j\}, \quad \text{if } k > \frac{n}{2}.$$

Proof. Let $V_k = \{1, 2, \ldots, k\}$ and let $\overline{V_k} = \{k+1, k+2, \ldots, n\}$, A triple T is an $(i, 3-i)$ triple if $|T \cap V_k| = i$ and $|T \cap \overline{V_k}| = 3 - i$. Let N_i be the number of $(i, 3-i)$ triples, $i = 0, 1, 2, 3$. Also let $N_i(x)$ be the number of

$(i, 3-i)$ triples that contain the vertex x. Summing $N_i(x)$ over all $x \in V_k$ counts each $(i, 3-i)$ triple i times, thus for $i = 0, 1, 2, 3$ we have

$$\sum_{x \in V_k} N_i(x) = i \cdot N_i. \tag{1}$$

Similarly, summing over $y \in \overline{V_k}$ we obtain for $i = 0, 1, 2, 3$

$$\sum_{y \in \overline{V_k}} N_i(y) = (3-i) \cdot N_i. \tag{2}$$

The number of points of intersection with triples and V_k is

$$\begin{aligned}
\sum_{x \in V_k} d_x &= 3N_3 + 2N_2 + N_1 = 3N_3 + \frac{3}{2}N_2 + \frac{1}{2}N_2 + N_1 \\
&= 3N_3 + \frac{3}{2}N_2 + \sum_{y \in \overline{V_k}} \left(\frac{1}{2}N_2(y) + \frac{1}{2}N_1(y) \right) \\
&= 3N_3 + \frac{3}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V_k}} (N_2(y) + N_1(y))
\end{aligned}$$

This last follows from Equation 2. For $y \in \overline{V_k}$, we have

$$N_2(y) + N_1(y) \leq N_2(y) + N_1(y) + N_0(y) = d_y.$$

Counting the number of points in V_k that are in triples that contain y we see that

$$N_2(y) + N_1(y) \leq 2N_2(y) + N_1(y) \leq k,$$

because each type $(i, 3-i)$ triple contains i points of V_k and any two triples that contain a fixed point y cannot intersect in another point. Thus

$$\sum_{x \in V_k} d_x \leq 3N_3 + \frac{3}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V_k}} \min\{d_y, k\}$$

Every type $(2,1)$ triple contains one of the $\binom{k}{2}$ possible pairs in V_k and every type $(3,0)$ contains 3. In a PSTS no pair is covered twice, thus $3N_3 + N_2 \leq \binom{k}{2}$, and hence

$$\sum_{x \in V_k} d_x \leq \binom{k}{2} + \frac{1}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V_k}} \min\{d_y, k\}.$$

Now for $y \in \overline{V_k}$ we have $N_2(y) \leq \lfloor \frac{k}{2} \rfloor$, so summing over $y \in \overline{V_k}$ we see that Equation 2 gives $N_2 \leq (n-k)\lfloor k/2 \rfloor$. For $x \in V_k$ we have $N_2(x) \leq (k-1)$. Thus using Equation 1 to sum over $x \in V_k$ we obtain $2N_2 \leq k(k-1)$. Consequently

$$N_2 \leq \min\{\frac{k(k-1)}{2}, (n-k)\left\lfloor \frac{k}{2} \right\rfloor\} = \begin{cases} \binom{k}{2}, & \text{if } k \leq \frac{n}{2} \\ (n-k)\left\lfloor \frac{k}{2} \right\rfloor, & \text{if } k > \frac{n}{2} \end{cases}$$

These 2 observations yield

$$\sum_{x \in V_k} d_x \quad \leq \quad \begin{cases} \frac{3}{2}\binom{k}{2} + \frac{1}{2} \sum_{y \in \overline{V_k}} \min\{k, d_y\}, & \text{if } k \leq \frac{n}{2} \\ \binom{k}{2} + \frac{1}{2}(n-k)\left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \sum_{y \in \overline{V_k}} \min\{k, d_y\}, & \text{if } k > \frac{n}{2}. \end{cases}$$

□

We conjecture that the conditions in the theorem are also sufficient. It should be noted that the condition obtained when $k = 1$ is that $2d_1 < n$. This is obviously necessary as the triangles containing a given point must otherwise be disjoint. In [1] the authors show that $2r < n$ and $rn \equiv 0 \pmod 3$ are necessary and sufficient for the existence of a partial Steiner triple system with degree sequence $(\underbrace{r, r, r, \ldots, r}_{n \text{ times}})$. This latter result also follows from the results in [2]. A partial Steiner triple system is said to be *equitable* if $|d_x - d_y| \leq 1$ for any two points x and y. In [2] it is shown that if there exists a partial Steiner triple system of order n with b triples, then there exists an equitable partial Steiner triple system of order n with b triples. Thus, by taking a maximum packing of triples on n points, deleting the appropriate number of triples, and applying this result, one can obtain a partial Steiner triple system in which all the vertices have degree r, whenever $2r < n$ and $rn = 0 \pmod 3$.

References

[1] M.S. Keranen, D.L. Kreher, W.H. Kocay and B. Li, Regular partial triple systems, *preprint.*

[2] L.D. Andersen, A.J.W Hilton and E. Mendelsohn, Embedding partial Steiner triple systems, Proc. London Math. Soc. (3) 41 (1980), 557–576.