Non-Fano Quads in Finite Projective Planes

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Abstract

Given a finite projective plane of order n. A quadrangle consists of four points A, B, C, D, no three collinear. If the diagonal points are non-collinear, the quadrangle is called a non-Fano quad. A general sum of squares theorem is proved for the distribution of points in a plane containing a non-Fano quad, whenever $n \geq 7$. The theorem implies that the number of possible distributions of points in a plane of order n is bounded for all $n \geq 7$. This is used to give a simple combinatorial proof of the uniqueness of $PP(7)$.

1 Introduction

A finite projective plane of order n is denoted $PP(n)$. A *quadrangle* (which we abbreviate to *quad*), is a set of four points, A, B, C, D, no three of which are collinear. The intersections of the lines AB and CD, AC and BD, and AD and BC determine three points E, F and G, respectively, called the *diagonal points* of the quad. If E, F and G are collinear, the quad is called a *Fano quad*. If they are non-collinear, we have a *non-Fano quad*.

We begin by counting the points and lines of $PP(n)$ in relation to a non-Fano quad A, B, C, D. We then consider a special distribution of points and lines which satisfies all pair-counts for all $n \geq 7$. In section 2, a sum-of-squares theorem is proved, describing the possible distributions of points in $PP(n)$, when $n \geq 7$. In section 3, all possible solution patterns are

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enumerated. There are only 54 possible solution patterns, for each $n \geq 7$. In section 4, the results of section 3 are used to give a simple combinatorial proof of the uniqueness of $PP(7)$.

Let A, B, C, D be a non-Fano quad in $PP(n)$, where $n \geq 7$. We then have three additional lines, EF, EG and FG, so that 9 lines are determined. These 9 lines containing the quad can be diagrammed as shown in Figure 1. The line EF must intersect the lines AD and BC, so that points u and v are determined. Similarly points w, x, y, z are determined by the requirement that the lines EG and FG must intersect each of the first 6 lines. This gives 9 lines of the plane, with 4 known points on each line. Since each line of $PP(n)$ has $n+1$ points, there are an additional $n-3$ points on each of these lines. We call these additional points *letters*. The letters on these 9 lines must all be distinct, because these 9 lines all intersect in exactly one point. Thus we have $9(n-3)$ letters. We also have the 7 points A, B, C, D, E, F, G, and the 6 points u, v, w, x, y, z, giving $9(n-3) + 7 + 6 = 9n - 14$ points. Now $PP(n)$ has $n^2 + n + 1$ points in total, so that there are an additional $n^2 - 8n + 15 = (n-3)(n-5)$ points, which we call *residual* points, none of which appear on the 9 lines we have so far constructed. We call these 9 lines the "quad" lines.

At this point it is convenient to colour the points according to their type. The quad points $A, B, \ldots G$ are coloured blue. The points u, v, w, x, y, z are coloured yellow. The letters are coloured green. The residual points are coloured red. We have four kinds of points in the projective plane, with respect to a given non-Fano quad.

ABEy **ACFw** **ADGu** **BCGv** **BDFx** **CDEz** **EFuv** **EGwx** **FGyz**	**9(n-3)** **letters**

Figure 1: A non-Fano quad in $PP(n)$

Each point of $PP(n)$ appears on $n+1$ lines. Since A, B, C, D have already appeared on 3 lines each, they must still occur on $n-2$ more lines. Similarly, E, F, G must each appear on an additional $n-3$ lines. We call these lines the "letter" lines. There are $4 \cdot (n-2) + 3 \cdot (n-3) = 7n - 17$ letter lines in total. Each residual point must appear exactly once in the letter lines containing each of $A, B, \ldots G$. Refer to Figure 2.

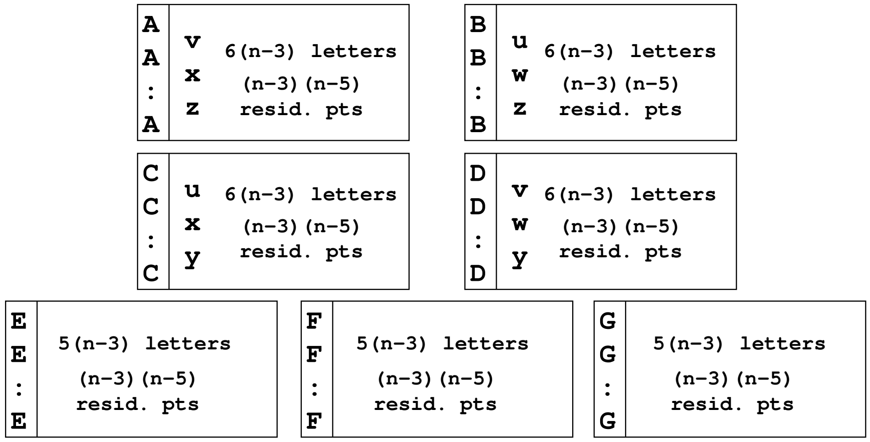

Figure 2: The "letter" lines in $PP(n)$.

Consider the letter lines containing E. Call them the E-lines. Since E has appeared 4 times in the quad lines, there are 5 quad lines not containing E. These 5 lines contain $5(n-3)$ letters, which must occur on the E-lines. The E-lines must also contain the $(n-3)(n-5)$ residual points. Now each quad line not containing E has $n-3$ letters, and these must occur on different E-lines. There are exactly $(n-3)$ E-lines. Since there are 5 quad lines not containing E, we conclude that each E-line must contain exactly 5 letters, and therefore exactly $n-5$ residual points. So the E-lines must look like Figure 3. The same applies to the F-lines and G-lines.

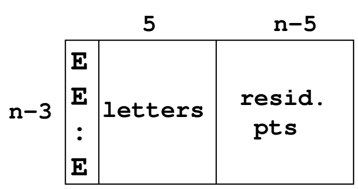

Figure 3: The E-lines in $PP(n)$.

Consider now the letter lines containing A. Call them the A-lines. A has appeared 3 times in the quad lines, so that there are 6 quad lines not containing A. These 6 lines contain $6(n-3)$ letters, which must occur in the A-lines. There can be at most 6 letters per A-line, as there are only 6 quad lines not containing A. The A-lines must also contain the $(n-3)(n-5)$ residual points. The quad lines containing A also contain the points u, w, y. Therefore v, x, z must each occur exactly one somewhere in the A-lines. This is indicated in Figure 2. A similar argument applies to the points occurring in the B-lines, C-lines, and D-lines. Since there are $n-3$ E-lines, each with $n-5$ residual points, there can be at most $n-3$ residual points on any A-line – for if there were $n-2$ or more, some pair of them would also occur together on an E-line, which is not possible. Two possible arrangements of points in the A-lines are shown in Figure 4. There may be other possible arrangements.

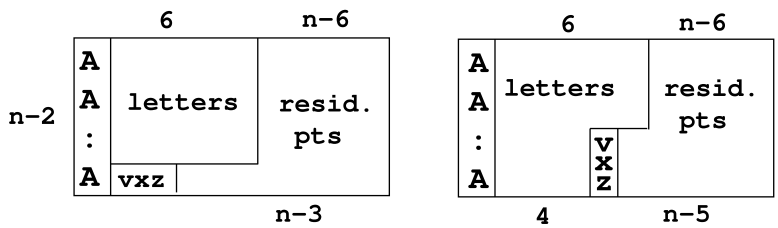

Figure 4: Two arrangements of the A-lines in $PP(n)$.

We now have $7n-17$ letter lines, and 9 quad lines, giving $7n-8$ lines. Since $PP(n)$ has n^2+n+1 lines, there are $n^2-6n+9=(n-3)^2$ more lines. We call these *residual* lines. Each residual point has occurred with each of the points A, B, C, D, E, F, G, i.e., 7 times so far. It follows that each residual point occurs $n-6$ times in the residual lines, giving $(n-3)(n-5)(n-6)$ occurrences there. The yellow points u, v, w, x, y, z have each appeared 4 times so far, and so must appear $n-3$ times in the residual lines, giving $6(n-3)$ occurrences there. The remaining points in the residual lines are letters, which occur a total of $(n+1)(n-3)^2 - (n-3)(n-5)(n-6) - 6(n-3) = 3(n-3)(3n-13)$ occurrences. There are several possible distributions of the points in the residual lines. One possible distribution is shown in Figure 5. We count the occurrences of the various types of points to confirm that the total counts are correct. We have $6(n-3)$ lines containing one yellow point, 7 letters, and $n-7$ residual points; and $(n-3)(n-9)$ lines containing 9 letters and $n-8$ residual points. This accounts for $6(n-3)$ occurrences of yellow points; $7\cdot 6(n-3)+9\cdot(n-3)(n-9) = 3(n-3)(3n+13)$ occurrences of letters; and $6(n-3)(n-7)+(n-3)(n-9)(n-8) = (n-3)(n-5)(n-6)$ occurrences of residual points. Notice that this distribution requires that $n \geq 9$ in order to make all the above counts non-negative.

Another distribution is shown in Figure 6. Here we have 4 lines each containing three yellow points, 3 letters, and $n-5$ residual points; $6(n-5)$ lines each containing 1 yellow point, 7 letters, and $n-7$ residual points;

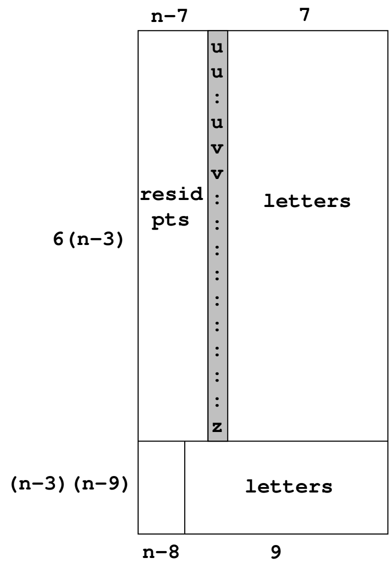

Figure 5: An arrangement of the residual lines in $PP(n)$, where $n \geq 9$.

and $(n-5)(n-7)$ lines containing 9 letters and $n-8$ residual points. We count the total number of occurrences of each type of point in order to confirm that the total counts are correct. Yellow points: there are $4 \cdot 3 + 6(n-5) = 6(n-3)$ occurrences in total; letters: there are $4 \cdot 3 + 7 \cdot 6(n-5) + 9(n-5)(n-7) = 3(n-3)(3n-13)$ occurrences; residual points: there are $4(n-5)+6(n-5)(n-7)+(n-5)(n-7)(n-8) = (n-3)(n-5)(n-6)$ occurrences in total. This distribution requires $n \geq 7$.

We summarise the numbers of the various types of points and lines in the following table. The table shows the number of each type of point, and the frequency of each type in the residual lines.

2 The Sum of Squares Theorem

We now consider the occurrences of pairs. The goal is to find all possible distributions of points which satisfy all pair counts. Such a distribution is called a *solution pattern*. Each pair of points must occur in exactly one line of the plane. All possible pairs of the points A, B, C, D, E, F, G occur in the quad lines. The $9(n-3)$ letters determine $\binom{9(n-3)}{2}$ pairs, of which $9\binom{n-3}{2}$ appear in the quad lines. The $(n-3)(n-5)$ residual points

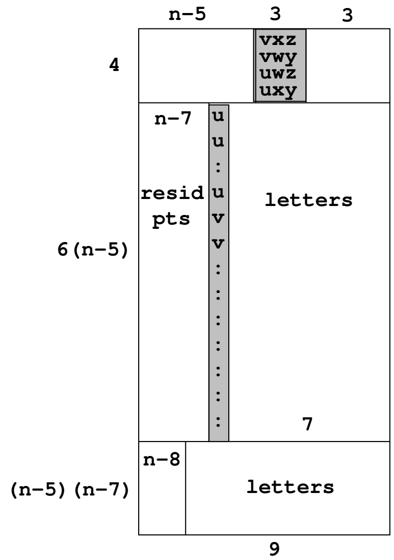

Figure 6: An arrangement of the residual lines in $PP(n)$, where $n \geq 7$.

determine $\binom{(n-3)(n-5)}{2}$ pairs, of which $3(n-3)\binom{n-5}{2}$ appear in the E, F, and G-lines. The pair counts of residual points and letter points in the A, B, C, D-lines, and in the residual lines, depend on the distribution of points. Let us choose the distribution of points in the A, B, C, D-lines to be that of the right diagram in Figure 4; and in the residual lines as in Figure 6, and count the pairs.

letter-letter pairs

$9\binom{n-3}{2}$ in the quad lines,
$+\ 4\{(n-5)\cdot\binom{6}{2}+3\binom{4}{2}\}$ in the A, B, C, D-lines,
$+\ 3(n-3)\cdot\binom{5}{2}$ in the E, F, G-lines,
$+\ 4\binom{3}{2}+6(n-5)\binom{7}{2}+(n-5)(n-7)\binom{9}{2}$ in the residual lines,
$=\binom{9(n-3)}{2}$

letter-residual point pairs

$4\cdot\{6(n-6)(n-5)+3\cdot 4(n-5)\}$ in the A, B, C, D-lines,
$+\ 3\cdot 5(n-5)(n-3)$ in the E, F, G-lines,
$+\ 4\cdot 3(n-5)+6(n-5)\cdot 7(n-7)+(n-5)(n-7)\cdot 9(n-8)$ in the residual lines
$=9(n-3)(n-3)(n-5)$

letter-$\{u, v, w, x, y, z\}$ pairs

type of point	number	res-line-freq
quad point	7	0
letters	$9(n-3)$	$(n-4)$ or $(n-5)$
u, v, w, x, y, z	6	$(n-3)$
residual	$(n-3)(n-5)$	$(n-6)$
type of line	**number**	
quad line	9	
letter line	$4(n-2)+3(n-3)$	
residual	$(n-3)^2$	

Figure 7: The types of points and lines in $PP(n)$.

$3 \cdot 2(n-3) + 6 \cdot 1(n-3)$ in the quad lines,
$+\ 4 \cdot 4 \cdot 3$ in the A, B, C, D-lines,
$+\ 4 \cdot 4 \cdot 3 + 1 \cdot 7 \cdot 6(n-5)$ in the residual lines,
$= 6 \cdot 9(n-3)$

residual point-residual point pairs

$4 \cdot \{(n-5)\binom{n-6}{2} + 3\binom{n-5}{2}\}$ in the A, B, C, D-lines,
$+\ 3 \cdot (n-3)\binom{n-5}{2}$ in the E, F, G-lines,
$+\ 4\binom{n-5}{2} + 6(n-5)\binom{n-7}{2} + (n-5)(n-7)\binom{n-8}{2}$ in the residual lines,
$= \binom{(n-3)(n-5)}{2}$

residual point-$\{u, v, w, x, y, z\}$ pairs

$4 \cdot 3(n-5)$ in the letter lines,
$+\ 4 \cdot 3(n-5) + 6(n-5)(n-7)$ in the residual lines,
$= 6(n-3)(n-5)$

$\{u, v, w, x, y, z\}$**-**$\{u, v, w, x, y, z\}$ **pairs**

3 in the quad lines
$+\ 4 \cdot \binom{3}{2}$ in the residual lines
$= \binom{6}{2}$

Thus, this distribution of points satisfies all the pair counts. We state this as a lemma.

Lemma 1 *The distribution of points given by Figures 1, 3, 6, and the right diagram of Figure 4 satisfy all pair counts, for* $n \geq 7$.

We now proceed to find all distributions that satisfy the pair counts. It will turn out that the number of allowable distributions is limited, for all n. We number the residual lines in Figure 6 from 1 to N, where $N = (n-3)^2$, and let r_i denote the number of letters in line i, where $i = 1, \ldots, N$.

According to Figure 6, $r_1, r_2, r_3, r_4 = 3$, $r_5, r_6, \ldots, r_M = 7$, where $M = 4 + 6(n-5)$, and $r_{M+1}, \ldots, r_N = 9$ is one possibility satisfying all the pair counts. We can always assume that $r_1 \leq r_2 \leq \ldots \leq r_N$. We let x_i denote the number of yellow points occurring in the i^{th} residual line. Then $(n+1) - r_i - x_i$ is the number of residual points occurring in the i^{th} residual line. We number the A-lines 1 to $n-2$, and let a_i denote the number of letters in line i, where $i = 1, \ldots, n-2$. One possibility is $a_i = 6$, for $i = 1, \ldots, n-5$ and $a_{n-4} = a_{n-3} = a_{n-2} = 4$. We can always assume that $a_1 \geq a_2 \geq \ldots \geq a_{n-2}$. We let α_i denote the number of yellow points occurring in the i^{th} A-line. Then $n - a_i - \alpha_i$ is the number of residual points occurring in the i^{th} A-line. Similarly b_i and β_i denote the number of letters and yellow points occurring in the B-lines; c_i and γ_i are for the C-lines; and d_i and δ_i are for the D-lines.

Counting the occurrences of the letter-letter pairs, yellow-yellow pairs, and red-red pairs, we obtain the following formulas.

$$\sum_{k=1}^{N} \binom{r_k}{2} + \sum_{k=1}^{n-2} \binom{a_k}{2} + \binom{b_k}{2} + \binom{c_k}{2} + \binom{d_k}{2} = 6(n-3)(6n-23) \quad (1)$$

Here $6(n-3)(6n-23)$ is obtained as $\binom{9(n-3)}{2} - 9\binom{n-3}{2} - 3(n-3)\binom{5}{2}$.

$$\sum_{k=1}^{N} \binom{x_k}{2} + \sum_{k=1}^{n-2} \binom{\alpha_k}{2} + \binom{\beta_k}{2} + \binom{\gamma_k}{2} + \binom{\delta_k}{2} = 12 \quad (2)$$

Here 12 is obtained as $\binom{6}{2} - 3$.

$$\sum_{k=1}^{N} \binom{(n+1) - r_k - x_k}{2} + \sum_{k=1}^{n-2} \binom{n - a_k - \alpha_k}{2} + \binom{n - b_k - \beta_k}{2} +$$

$$+ \sum_{k=1}^{n-2} \binom{n - c_k - \gamma_k}{2} + \binom{n - d_k - \delta_k}{2} = \frac{1}{2}(n-3)(n-5)(n^2 - 11n + 32) \quad (3)$$

Here $\frac{1}{2}(n-3)(n-5)(n^2 - 11n + 32)$ is obtained as $\binom{(n-3)(n-5)}{2} - 3(n-3)\binom{n-5}{2}$.

There are three additional equations that can be obtained by counting the number of pairs composed of two different kinds of points (eg., letter and residual point, etc.), but the resulting equations are not independent. It is sufficient to consider the above three equations.

In order to simplify the equations, we do the following. Write:

$$
\begin{array}{lll}
r_k = 3 + \delta r_k; & x_k = 3 + \delta x_k; & \text{for } k = 1, \dots, 4 \\
r_k = 7 + \delta r_k; & x_k = 1 + \delta x_k; & \text{for } k = 5, \dots, M \\
r_k = 9 + \delta r_k; & x_k = 0 + \delta x_k; & \text{for } k = M+1, \dots, N
\end{array}
$$

$$
\begin{array}{lll}
a_k = 6 + \delta a_k; & b_k = 6 + \delta b_k; & \text{for } k = 1, \dots, n-5 \\
a_k = 4 + \delta a_k; & b_k = 4 + \delta b_k; & \text{for } k = n-4, n-3, n-2 \\
c_k = 6 + \delta c_k; & d_k = 6 + \delta d_k; & \text{for for } k = 1, \dots, n-5 \\
c_k = 4 + \delta c_k; & d_k = 4 + \delta d_k; & \text{for } k = n-4, n-3, n-2
\end{array}
$$

$$
\begin{array}{lll}
\alpha_k = 0 + \delta \alpha_k; & \beta_k = 0 + \delta \beta_k; & \text{for } k = 1, \dots, n-5 \\
\alpha_k = 1 + \delta \alpha_k; & \beta_k = 1 + \delta \beta_k; & \text{for } k = n-4, n-3, n-2 \\
\gamma_k = 0 + \delta \gamma_k; & \delta_k = 0 + \delta \delta_k; & \text{for for } k = 1, \dots, n-5 \\
\gamma_k = 1 + \delta \gamma_k; & \delta_k = 1 + \delta \delta_k; & \text{for } k = n-4, n-3, n-2
\end{array}
$$

Counting the occurrences of the various points then gives the following formulas.

$$
\sum_{k=1}^{N} \delta r_k = 0, \qquad \sum_{k=1}^{N} \delta x_k = 0
$$

$$
\sum_{k=1}^{n-2} \delta a_k = 0, \qquad \sum_{k=1}^{n-2} \delta b_k = 0, \qquad \sum_{k=1}^{n-2} \delta c_k = 0, \qquad \sum_{k=1}^{n-2} \delta d_k = 0
$$

$$
\sum_{k=1}^{n-2} \delta \alpha_k = 0, \qquad \sum_{k=1}^{n-2} \delta \beta_k = 0, \qquad \sum_{k=1}^{n-2} \delta \gamma_k = 0, \qquad \sum_{k=1}^{n-2} \delta \delta_k = 0 \qquad (*)
$$

Substituting for r_k, a_k, b_k, c_k and d_k into formula (1), we obtain

$$
\sum_{k=1}^{4} \binom{3 + \delta r_k}{2} + \sum_{k=5}^{M} \binom{7 + \delta r_k}{2} + \sum_{k=M+1}^{N} \binom{9 + \delta r_k}{2} +
$$

$$
+ \sum_{k=1}^{n-5} \binom{6 + \delta a_k}{2} + \binom{6 + \delta b_k}{2} + \binom{6 + \delta c_k}{2} + \binom{6 + \delta d_k}{2} +
$$

$$
+ \sum_{k=n-4}^{n-2} \binom{4 + \delta a_k}{2} + \binom{4 + \delta b_k}{2} + \binom{4 + \delta c_k}{2} + \binom{4 + \delta d_k}{2} =
$$

$$
= 6(n-3)(6n-23)
$$

Writing $\binom{m}{2} = m(m-1)/2$, and using the summations $(*)$, this expression can be expanded and simplified to:

$$\sum_{k=1}^{N} \delta r_k^2 + \sum_{k=1}^{n-2} (\delta a_k^2 + \delta b_k^2 + \delta c_k^2 + \delta d_k^2) =$$

$$4 \sum_{k=n-4}^{n-2} (\delta a_k + \delta b_k + \delta c_k + \delta d_k) - 8 \sum_{k=5}^{M} \delta r_k - 12 \sum_{k=M+1}^{N} \delta r_k \tag{4}$$

Substituting into formulas (2) and (3) produces the following results:

$$\sum_{k=1}^{N} \delta x_k^2 + \sum_{k=1}^{n-2} (\delta \alpha_k^2 + \delta \beta_k^2 + \delta \gamma_k^2 + \delta \delta_k^2) =$$

$$-2 \sum_{k=n-4}^{n-2} (\delta \alpha_k + \delta \beta_k + \delta \gamma_k + \delta \delta_k) + 4 \sum_{k=5}^{M} \delta x_k + 6 \sum_{k=M+1}^{N} \delta x_k \tag{5}$$

$$\sum_{k=1}^{N} (\delta r_k + \delta x_k)^2 +$$

$$+ \sum_{k=1}^{n-2} [(\delta a_k + \delta \alpha_k)^2 + (\delta b_k + \delta \beta_k)^2] + (\delta c_k + \delta \gamma_k)^2 + (\delta d_k + \delta \delta_k)^2] =$$

$$= 2 \sum_{k=n-4}^{n-2} (\delta a_k + \delta \alpha_k + \delta b_k + \delta \beta_k + \delta c_k + \delta \gamma_k + \delta d_k + \delta \delta_k) +$$

$$-4 \sum_{k=5}^{M} (\delta r_k + \delta x_k) - 6 \sum_{k=M+1}^{N} (\delta r_k + \delta x_k) \tag{6}$$

Now we use formula (4) to solve for $4 \sum_{k=5}^{M} \delta r_k + 6 \sum_{k=M+1}^{N} \delta r_k$, and formula (5) to solve for $4 \sum_{k=5}^{M} \delta x_k + 6 \sum_{k=M+1}^{N} \delta x_k$, which we substitute into formula (6). The result is:

$$\sum_{k=1}^{N} (\delta r_k + \delta x_k)^2 + \sum_{k=1}^{n-2} [(\delta a_k + \delta \alpha_k)^2 + (\delta b_k + \delta \beta_k)^2 + (\delta c_k + \delta \gamma_k)^2 + (\delta d_k + \delta \delta_k)^2] =$$

$$= 2 \sum_{k=n-4}^{n-2} (\delta a_k + \delta \alpha_k + \delta b_k + \delta \beta_k + \delta c_k + \delta \gamma_k + \delta d_k + \delta \delta_k) +$$

$$+\frac{1}{2}\sum_{k=1}^{N}\delta r_k^2+\frac{1}{2}\sum_{k=1}^{n-2}(\delta a_k^2+\delta b_k^2+\delta c_k^2+\delta d_k^2)-2\sum_{k=n-4}^{n-2}(\delta a_k+\delta b_k+\delta c_k+\delta d_k)+$$

$$-\sum_{k=1}^{N}\delta x_k^2-\sum_{k=1}^{n-2}(\delta\alpha_k^2+\delta\beta_k^2+\delta\gamma_k^2+\delta\delta_k^2)-2\sum_{k=n-4}^{n-2}(\delta\alpha_k+\delta\beta_k+\delta\gamma_k+\delta\delta_k)$$

A number of terms cancel, leaving

$$\sum_{k=1}^{N}(\delta r_k+\delta x_k)^2+\sum_{k=1}^{n-2}[(\delta a_k+\delta\alpha_k)^2+(\delta b_k+\delta\beta_k)^2+(\delta c_k+\delta\gamma_k)^2+(\delta d_k+\delta\delta_k)^2]=$$

$$\frac{1}{2}\sum_{k=1}^{N}\delta r_k^2+\frac{1}{2}\sum_{k=1}^{n-2}[(\delta a_k^2+\delta b_k^2+\delta c_k^2+\delta d_k^2)]-\sum_{k=1}^{N}\delta x_k^2-\sum_{k=1}^{n-2}(\delta\alpha_k^2+\delta\beta_k^2+\delta\gamma_k^2+\delta\delta_k^2)$$

Expanding the squares on the left-hand side, and moving the right-hand side to the left leaves

$$\sum_{k=1}^{N}(\frac{1}{2}\delta r_k^2+2\delta r_k\delta x_k+2\delta x_k^2)+$$

$$+\sum_{k=1}^{n-2}(\frac{1}{2}\delta a_k^2+2\delta a_k\delta\alpha_k+2\delta\alpha_k^2+\frac{1}{2}\delta b_k^2+2\delta b_k\delta\beta_k+2\delta\beta_k^2)+$$

$$+\sum_{k=1}^{n-2}(\frac{1}{2}\delta c_k^2+2\delta c_k\delta\gamma+2\delta\gamma_k^2+\frac{1}{2}\delta d_k^2+2\delta d_k\delta\delta_k+2\delta\delta_k^2)=0$$

Rewriting this gives the following

Theorem 1 *Let the quantities $\delta r_k, \delta x_k, \delta a_k, \delta b_k, \delta c_k, \delta d_k, \delta\alpha_k, \delta\beta_k, \delta\gamma_k, \delta\delta_k$ be as above. Then*

$$\sum_{k=1}^{N}(\delta r_k+2\delta x_k)^2+$$

$$+\sum_{k=1}^{n-2}[(\delta a_k+2\delta\alpha_k)^2+(\delta b_k+2\delta\beta_k)^2+(\delta c_k+2\delta\gamma_k)^2+(\delta d_k+2\delta\delta_k)^2]=0$$

Corollary 1 *The δr_k and δx_k satisfy $\delta r_k=-2\delta x_k$, for $k=1,\ldots,N$. The $\delta a_k, \delta b_k, \delta c_k, \delta d_k$ and $\delta\alpha_k, \delta\beta_k, \delta\gamma_k, \delta\delta_k$ satisfy $\delta a_k=-2\delta\alpha_k, \delta b_k=-2\delta\beta_k, \delta c_k=-2\delta\gamma_k, \delta d_k=-2\delta\delta_k$, for $k=1,\ldots,n-2$.*

We will use this corollary to completely determine all possible distribution patterns of points in a projective plane $PP(n)$ containing a non-Fano quad, where $n \geq 7$. We shall see that there are only 9 basic solution patterns possible. Now we have the freedom of ordering the r_k so that $r_1 \leq r_2 \leq \ldots \leq r_N$. Similarly, we can take $a_1 \geq a_2 \geq \ldots \geq a_{n-2}$; $b_1 \geq b_2 \geq \ldots \geq b_{n-2}$; $c_1 \geq c_2 \geq \ldots \geq c_{n-2}$; and $d_1 \geq d_2 \geq \ldots \geq d_{n-2}$. We also know that each $r_k \leq 9$, since the quad lines are 9 lines which contain 7 letters each. Any residual line with more than 9 letters would necessarily have a pair that also occurs in a quad line.

Since $\delta r_k = -2\delta x_k$, and since the initial solution has $r_k = 3, 7$ or 9, it follows that all r_k are odd numbers, and that all $r_k \leq 9$. Similarly, we conclude that all a_k, b_k, c_k, d_k are even numbers ≤ 6. This gives the following:

Corollary 2 *There are only three possible patterns for the solutions for the A, B, C and D-lines.*

The three possible patterns are illustrated in Figure 8. We say these solution patterns are of type $1, 2$, or 3 according to the maximum number of yellow points occurring in a line. We will see that the number of possible solution patterns for the residual lines is also very limited.

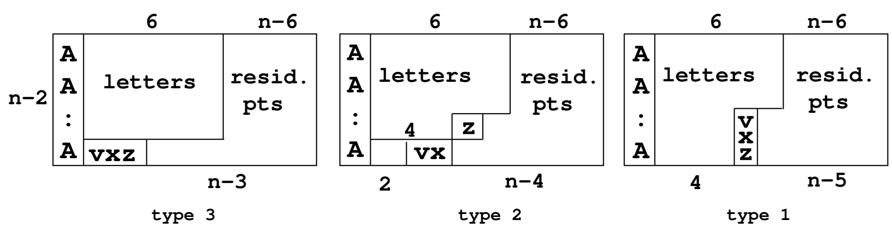

Figure 8: The possible A-line solution patterns

3 The Solutions

Consider the distribution of the yellow points $\{u, v, w, x, y, z\}$ in the A, B, C, D-lines, as shown in Figures 2 and 8. The yellow points occur in four groups of three: $\{v, x, z\}$ with the A-lines; $\{u, w, z\}$ with the B-lines; $\{u, x, y\}$ with the C-lines; and $\{v, w, y\}$ with the D-lines. We will call these four triples the A-triple, B-triple, C-triple, and D-triple, respectively. Notice that any two of these triples intersect in exactly one point. Furthermore, the only pairs of yellow points which appear in the quad lines are uv, wx, and yz, and these pairs never occur in any of the above four

Triples	Complements
$A: \; vxz$	uwy
$B: \; uwz$	vxy
$C: \; uxy$	vwz
$D: \; vwy$	uxz

Figure 9: The A, B, C, D-triples, and their complements

triples. It follows that for each triple, the three pairs corresponding to it must appear in the letter lines and/or residual lines. Since any 4-subset of the yellow points necessarily contains one of the pairs uv, wx, or yz, we conclude that any residual line can contain at most three yellow points, i.e., $x_i \leq 3$, for all i.

Corresponding to any triple, eg., the A-triple $\{v, x, z\}$, there is a complementary triple – in this case $\{u, w, y\}$, which doesn't correspond to any of A, B, C, D. The complementary triples are shown in Figure 9. The A, B, C and D-triples are called *primary* triples.

Lemma 2 *The distribution pattern of the yellow points completely determines the distribution pattern of the solution.*

Proof. There are 12 pairs of yellow points that do not appear in the quad lines. Suppose there are k residual lines containing three yellow points. Then $0 \leq k \leq 4$. These can be taken as the first k residual lines. This determines $3k$ pairs of yellow points. Some or all of the remaining $12 - 3k$ pairs appear in the A, B, C and D-lines, which must be of type $1, 2$ or 3. Any remaining pairs appear in the residual lines. The residual lines in which they appear have $x_i = 2$, and these can be taken as the $(k{+}1)^{\mathrm{st}}, (k{+}2)^{\mathrm{nd}}, \ldots$ residual lines. With respect to the solution of Figure 6, this determines δx_i for these lines. Since $\sum_i \delta x_i = 0$, this in turn determines the number of x_i, where $i \geq N = 6(n-5)+5$, that have $\delta x_i = +1$. The solution pattern is thereby completely determined.

The above lemma allows a complete determination of all possible solution patterns. We can use it to find all possible solutions for some small planes. There are 9 basic solution patterns, organized according to the number of residual lines with $x_i = 3$. In the next section, we will use these to prove the uniqueness of the plane of order 7. Each solution pattern can be further subdivided according to the distribution of yellow points in the A, B, C, and D-lines. With the small planes, some of these will not be possible. Some solution patterns are shown in Figures $10, 11, 12$, etc.

Solutions with no Complementary Triples

Suppose first that the residual lines contain no complementary triples. Then a residual line with $x_i = 3$ necessarily contains a primary triple. The corresponding letter line solutions must be of type 1.

1) (1 solution) Suppose there are 4 residual lines with $x_i = 3$. The distribution of yellow points in the residual lines must be as in Figure 6, and must be of type 1 in the letter lines (see Figure 8).

2) (3 solutions) Suppose there are 3 residual lines with $x_i = 3$. Without loss of generality, these can be taken to be the A, B, and C-triples, as shown in Figure 2. The D-lines solution can be of type $1, 2$ or 3 (see Figure 8), giving three solutions.

2a) If the D-lines solution is of type 1, the three D-pairs vw, vy, wy must occur on separate residual lines. We must have $\delta x_4 = -1, \delta x_5 = +1, \delta x_6 = +1$. Therefore $\delta x_N = -1$, where $N = 6(n-5)+5$. The left diagram in Figure 10 is the only possibility.

2b) If the D-lines solution is of type 2, then two of the D-pairs, say vw, vy, must occur on separate residual lines. We must have $\delta x_4 = -1$ and $\delta x_5 = +1$. The middle diagram in Figure 10 is the only possibility.

2c) If the D-lines solution is of type 3, the three D-pairs occur in the D-lines. We must have $\delta x_4 = -2, \delta x_N = +1$, and $\delta x_{N+1} = +1$. The right diagram in Figure 10 is the only possibility.

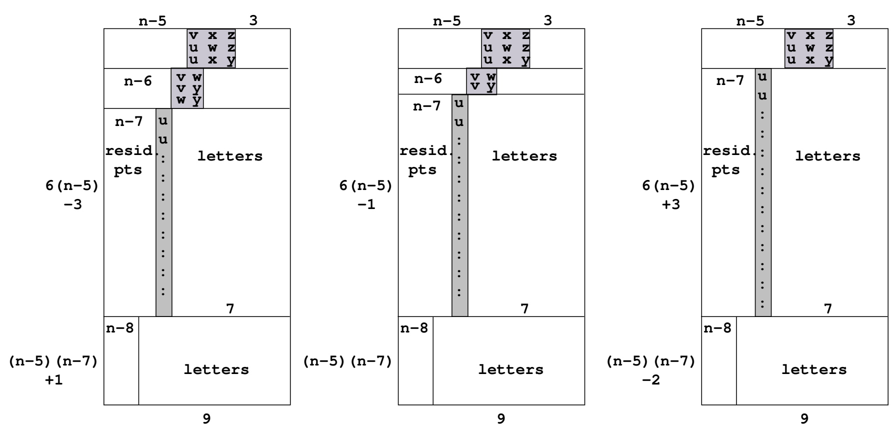

Figure 10: Solutions 2a,b,c

3) (6 solution patterns) Suppose there are two residual lines with $x_i = 3$. Without loss of generality, these can be taken to be the A and B-triples, as shown in Figure 2. This distinguishes point z, as it is the only point common to the A and B-triples. It also distinguishes point y, which is the only point common to the C and D-triples. The C and D-lines solutions can be of type $1, 2$ or 3 (see Figure 8). Up to symmetry this gives 6 solution patterns, with 10 possible solutions.

3a) If the C-lines solution is of type 1, the D-lines solution can be of type $1, 2$ or 3, as shown in Figure 11. In the situation when the D-lines solution is of type 2, there is a D-line containing only one yellow point. We must distinguish the cases when this point is y or not y. This results in two ways of filling in the spaces marked by $*$ in the diagram. In each case we use the δx_i's to determine the three possible solution patterns.

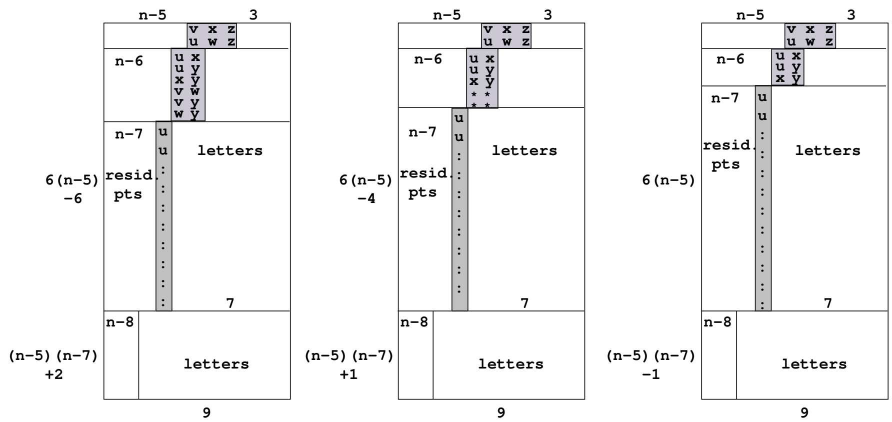

Figure 11: Solutions 3a

3b) If the C-lines solution is of type 2, we can assume that the D-lines solution is of type 2 or 3 (for type 1 reduces to case (3a), by interchanging C and D). There is a C-line containing only one yellow point. We must distinguish the cases when this point is y or not y. A similar situation holds when the D-lines solution is of type 2. The result is two solution patterns, shown in Figure 12. There are two ways of filling in the spaces marked by $*$ in the middle diagram, and 3 ways for the left diagram, giving 5 solutions, and two solution patterns.

3c) If the C-lines solution is of type 3, we can assume that the D-lines solution is also of type 3, The unique possibility is shown

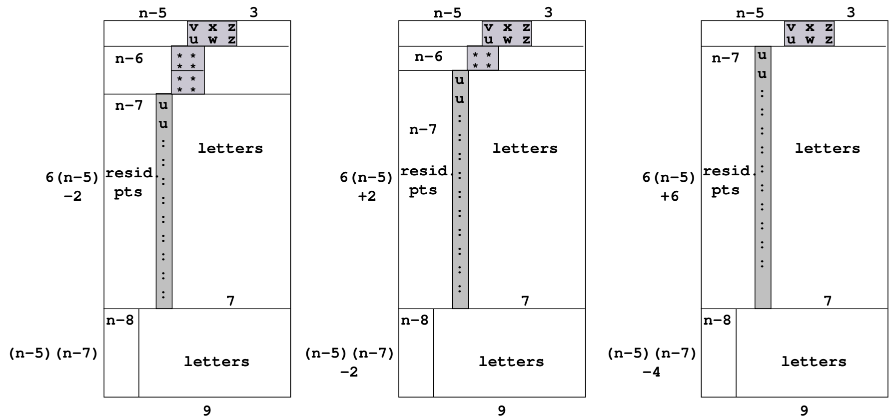

Figure 12: Solutions 3b and 3c

in the right diagram of Figure 12.

4) (9 solution patterns) Suppose there is one residual line with $x_i = 3$. Without loss of generality, this can be taken to be the A-triple, as shown in Figure 2. The B, C, and D-lines solutions can be of type 1, 2 or 3 (see Figure 8). The number of pairs of yellow points occurring in the B, C, and D-lines can then be $0, 1, 2, 3, 4, 5, 6, 7$ or 9. In each case, the number of residual lines with $x_i = 2$ is completely determined. This in turn completely determines the number of residual lines with $x_i = 1$. Refer to Figure 13.

5) (12 solution patterns) Suppose there are no residual lines with $x_i = 3$. The A, B, C, and D-lines solutions are of type 1, 2 or 3 (see Figure 2). The number of pairs of yellow points occurring in the A, B, C, and D-lines can then be $0, 1, \ldots, 10$ or 12. In each case, the number of residual lines with $x_i = 2$ is completely determined. This in turn completely determines the number of residual lines with $x_i = 1$. Refer to Figure 13.

We summarize the preceding results as:

Lemma 3 *If there are no complementary triples appearing in the residual lines, then there are 31 possible solution patterns.*

Solutions with Complementary Triples

If the residual lines with $x_i = 3$ contain complementary triples (see Figure 9), then a number of other solution patterns are possible. Observe

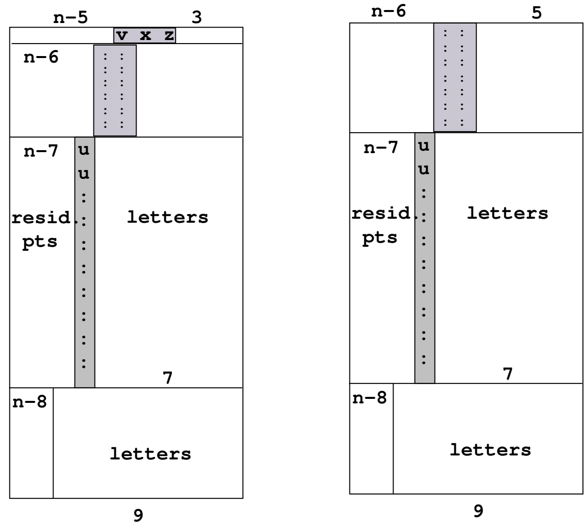

Figure 13: Solution patterns for cases 4 and 5

that this implies the existence of Fano quads in the plane. For example, the complementary A-triple is uwy. The diagonal points of the quad $\{E, F, G, A\}$ are u, w, y (see Figure 1). If these three points are collinear, then $\{E, F, G, A\}$ is a Fano quad.

6) (1 solution) Suppose there are 4 residual lines with $x_i = 3$, containing complementary triples. The distribution of yellow points in the residual lines must be as in Figure 6, except that the triples are now the complementary triples instead. The lettter lines must be of type 1 (see Figure 8).

7) (4 solution patterns) Suppose there are 3 residual lines with $x_i = 3$, containing complementary triples. Without loss of generality the triples can be taken as uwy, uxz and vwz. The B-lines must be of type 1, as the three pairs of the B-triple uwz occur in these complementary triples. The A, C and D-lines can be of type 1 or 2. This gives four possible solutions.

8) (5 solution patterns) Suppose there are 2 residual lines with $x_i = 3$, containing complementary triples. Without loss of generality the triples can be taken as uwy and uxz. Notice that every primary triple intersects one of uwy and uxz in a pair of points. Hence there cannot be any primary triples occurring. There are 6 pairs of yellow points which do not occur in triples, namely vx, vy, vw, vz, xy, wz. The A, B, C and D-lines must be of type 1 or 2, giving 5 possible solution patterns.

9) (13 solution patterns) Suppose there is 1 residual line with $x_i = 3$, containing a complementary triple. Without loss of generality the triple can be taken as uwy. The A-triple vxz is the only primary triple which does not intersect uwy in a pair. This gives two subcases.

- **9a)** There is a residual line with $x_i = 3$ containing vxz. The pairs of yellow points not occurring in the triples uwy and vxz are uz, wz, xy, vy. The A-lines must be of type 1. The B, C and D-triples can be of type 1 or 2, giving 4 possible solution patterns.
- **9b)** There is no residual line containing vxz. The A-lines can be of type $1, 2$ or 3. The B, C and D-lines can be of type 1 or 2. There are 4 solution patterns when the A-lines are of type 3. There are 5 solution patterns when the A-lines are of type 1 or 2.

We summarize the preceding results as:

Lemma 4 *If there are complementary triples appearing in the residual lines, then there are 23 possible solution patterns.*

In addition we make the following observation.

Lemma 5 *A plane $PP(n)$ in which the A, B, C and D-lines are all of type 3 contains a sub-plane of order 3.*

Proof. The points of the sub-plane are $A, B, C, D, E, F, G, u, v, w, x, y, z$; giving 13 points. The lines are the 4-subsets of the lines of $PP(n)$ induced by these points. There are 9 quad lines, each of which contains exactly 4 sub-plane points; and one line in each of the A, B, C, and D-lines.

4 Uniqueness of the Plane of Order 7

In this section we use the sum of squares theorem to prove the uniqueness of $PP(7)$. This can easily be proved by exhaustive enumeration by computer, but a simple combinatorial proof is also useful. A proof of the uniqueness originally appeared in three papers, in 1953 and 1954. In [6], Pierce proved that every quad in $PP(7)$ must be a non-Fano quad, by reducing the case of a Fano quad to the Kirkman Schoolgirl problem, and enumerating all possible solutions. In [2], Hall gave a proof of the uniqueness, building on

Pierce's result. He assumed that all quads are non-Fano quads, and then reduced the problem to a theorem of Moufang [4], which implied that the plane must be Desarguesian. Note that Moufang's paper is 65 pages long. Hall's paper contained an error, pointed out by G. Pickert. This error was corrected by Hall in [3]. In the book "Projektive Ebenen" [5] by Pickert, a proof following Hall is given (pp. 319-325) that if all quads of $PP(7)$ are non-Fano quads, then the plane must be Desarguesian. When a finite plane is known to be Desarguesian, we can use a deep result described in [1] (pp. 68-70) to conclude that the plane can be coordinatized by a field, and is therefore unique.

We use the sum of squares theorem to give an elementary combinatorial proof of the uniqueness of $PP(7)$. This will also imply that the automorphism group is transitive on all quads of $PP(7)$. Using Pierce's result [6], we can assume that all quads are non-Fano quads. $PP(7)$ contains 57 points and 57 lines of 8 points each. There are 36 letters and 8 residual points. There are 16 residual lines. Each residual point occurs exactly once in the residual lines. Refer to the table of Figure 7. Many of the possible solution patterns can be easily ruled out. We shall see that Case (1) is the only possible solution pattern and that the completion to a projective plane is unique. Let the residual points be $1, 2, 3, 4, 5, 6, 7, 8$. As there are only 8 occurrences of the residual points in the residual lines, we can arbitrarily number them with $1, 2, \ldots, 8$, and then determine the compatible distributions of the residual points in the letter lines.

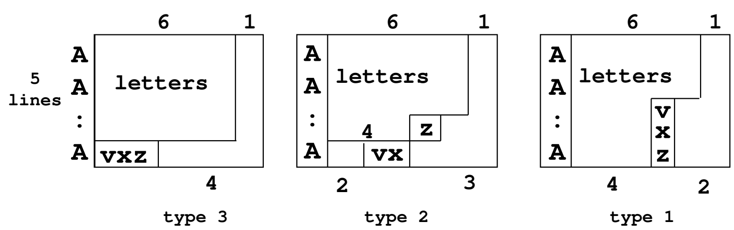

Figure 14: A-line solution patterns for $PP(7)$

Lemma 6 *Cases (2), (3) and (4) are not possible.*

Proof. Refer to Figures 10, 11, 12 and 13. Notice that cases (2a), (2c), (3a) and (3c) are not possible, because there are not 8 spaces for the residual points to occur in the residual lines when $n = 7$. The only possibilities are cases (2b) (middle diagram of Figure 10), (3b) (left diagram of Figure 12), and (4) (left diagram of Figure 13). In each case, there is a residual line containing the A-triple vxz, as well as residual lines containing two pairs of the D-triple vwy. This is illustrated in Figure 15, where the pair of yellow points occurring in the D-lines is vw; a similar argument also holds if it is vy or wy. Points 7 and 8 cannot occur in either of the last two D-lines, because they have already both occurred with two of v, w, y in the residual lines. Hence 7 and 8 must occur in the first two D-lines. Points 1 and 2 cannot occur together in the last two D-lines, and cannot occur at all in the D-line containing v. Hence, at most one of them can occur in the last two D-lines. Therefore one of them, say 2, occurs in the third D-line. Now any two primary triples intersect in a point. Therefore 3 and 4 have already both occurred with one of v, w, y in the residual lines. It follows that at most one of them can occur in the last two D-lines, say 3. But then there is no place for point 4, a contradiction.

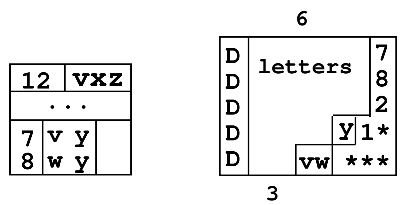

Figure 15: Cases (2), (3), and (4), $PP(7)$

Lemma 7 *Case (5) is not possible.*

Proof. Refer to Figures 13 (right diagram) and 16. There must be 8 residual lines containing a pair of yellow points. Without loss of generality, the pairs vz and xz occur in the first two residual lines, and vx occurs in the A-lines. Points 1 and 2 must occur in the first two A-lines. The C and D-triples intersect vxz in x and v, respectively. Therefore at most one of points 5 and 6 can occur in the last two A-lines, and at most one of points 7 and 8 can occur there. This is a contradiction.

Lemma 8 *Cases (6), (7), (8), and (9) are not possible.*

Proof. The existence of a line containing a complementary triple implies that the plane contains a Fano quad, which is impossible.

Note that if we do not rely on Pierce's result that all quads of $PP(7)$ must be non-Fano quads, we can easily prove that the cases of Lemma 8 are not possible, using the same methods as in Lemmas 6 and 7.

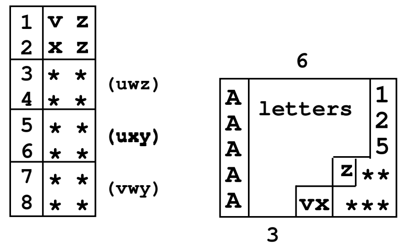

Figure 16: Case (5), $PP(7)$

The preceding lemmas give the following:

Theorem 2 *Every non-Fano quad in a $PP(7)$ belongs to solution pattern (1).*

We shall see that the distribution of yellow points completely determines the remaining structure of the plane. Figure 17 shows the first four residual lines of solution pattern 1. Here we have labelled the 12 letters occurring in these four lines $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3$.

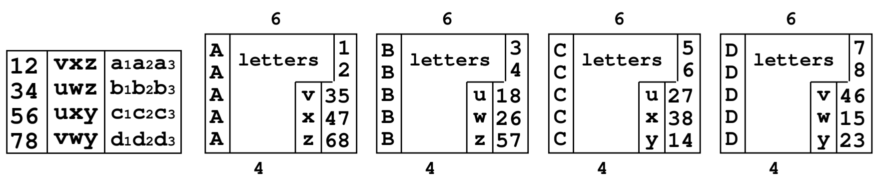

Figure 17: Case (1), $PP(7)$

Notice that points 1 and 2 must occur in the first two A-lines. Points 3 and 4 have already occurred with z. Hence they occur in the A-lines with v and x. Points 5 and 6 have already occurred with x. Hence they appear in the A-lines with v and z. This leaves two places for points 7 and 8. We then find that the residual points of the B, C and D-lines can be uniquely filled in.

We now find that of the $\binom{8}{2}$ possible pairs of residual points $1, 2, \ldots, 8$, the pairs which have not yet occurred form the graph shown in Figure 18. These remaining pairs must occur in the E, F and G-lines, which each contains exactly two residual points. So the E, F and G-lines must together contain a 1-factorization of this graph.

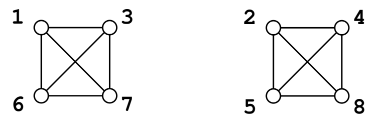

Figure 18: $PP(7)$, the remaining pairs of residual points

At this point we give names to the letters appearing in the last three quad lines, calling them $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4, g_1, g_2, g_3, g_4$. Refer to Figure 19. Notice that e_1, e_2, e_3, e_4 must appear in distinct E-lines; that f_1, f_2, f_3, f_4 must appear in distinct F-lines; and that g_1, g_2, g_3, g_4 must appear in distinct G-lines.

Consider now the letters a_1, a_2, a_3. They must occur in the quad lines. They occur with v, x, z in the residual lines (see Figure 17). This leaves only three possible lines in the quad lines where they may occur. Similarly for b_1, b_2, b_3 and c_1, c_2, c_3 and d_1, d_2, d_3. As the points have not yet been distinguished, their placement in the quad lines is unique – refer to Figure 19. Of these 12 letters, six of them, $b_2, b_3, c_2, c_3, d_2, d_3$ do not occur with A in the quad lines, and so must occur in the A-lines. It is easy to verify that they cannot occur in the lines containing v, x or z. Therefore they occur in the first two A-lines, with three occurring per line. Now b_2c_2 and c_3d_3 occur in quad lines. Hence, it must be the triples $b_2c_3d_2$ and $b_3c_2d_3$ which occur in the A-lines. At this point we do not as yet know which triple occurs with point 1, and which occurs with point 2. Assume for the time being that they are placed in the A-lines as in Figure 19.

ABEy	$a_1 b_1$
ACFw	$a_2 c_1$
ADGu	$a_3 d_1$
BCGv	$b_2 c_2$
BDFx	$b_3 d_2$
CDEz	$c_3 d_3$
EFuv	$g_1 g_2 g_3 g_4$
EGwx	$f_1 f_2 f_3 f_4$
FGyz	$e_1 e_2 e_3 e_4$

6

A	$b_2c_3d_2$		1
A	$b_3c_2d_3$		2
A		v	35
A		x	47
A		z	68

4

6

B	$a_3c_1d_3$		3
B	$a_2c_3d_1$		4
B		u	18
B		w	26
B		z	57

4

6

C	$a_3b_1d_2$		5
C	$a_1b_3d_1$		6
C		u	27
C		x	38
C		y	14

4

6

D	$a_2b_1c_2$		7
D	$a_1b_2c_1$		8
D		v	46
D		w	15
D		y	23

4

Figure 19: $PP(7)$

We now observe that b_2 has already occurred with the residual points $1, 3, 4$, and so must still occur with $2, 5, 6, 7, 8$. We also find that b_2 has occurred with points B, C, G in the quad lines, and so must appear once in the E-lines, once in the F-lines, and once in the D-lines. In the E and

F-lines, b_2 occurs in a line containing two residual points. Referring to Figure 18, we see that b_2 occurs with two points of $1, 3, 6, 7$ or two points of $2, 4, 5, 8$ when it occurs in the E and F-lines. Comparing this with the points $2, 5, 6, 7, 8$, we see that in the D-lines, b_2 must appear with point 8, and not 7. This implies that b_2 occurs with the pairs 25 and 67 in the E and F-lines.

We then turn to the D-lines. D must still occur with $a_1, a_2, b_1, b_2, c_1, c_2$, and these points must occur in the first two D-lines, in two triples. Since a_1b_1 and b_2c_2 have already occurred in the quad lines, we find that the distribution must be as shown in the D-lines of Figure 19. Continuing as in the above paragraph, we find that the distribution of the a_i, b_i, c_i and d_i in the A, B, C and D-lines is completely determined by the initial choice which placed b_2 in the first A-line. If we had placed it instead in the second A-line, the remaining distribution in the B, C and D-lines would also be forced. This second possibility is obtained by interchanging the triples in the first two rows of the A, B, C and D-lines.

The pairs of residual points that the a_i, b_i, c_i and d_i must occur with in the E, F and G-lines are then also forced. The result is shown in the following table, for each of the two possible distributions. As each possible pair of residual points occurs exactly twice in each half of the table, we conclude that the E, F, G-lines each contain exactly two of the a_i, b_i, c_i and d_i.

point	$\{1,3,6,7\}$	$\{2,4,5,8\}$	$\{1,3,6,7\}$	$\{2,4,5,8\}$
a_1	37	45	36	48
a_2	36	58	67	45
a_3	67	48	37	58
b_1	16	28	17	25
b_2	67	25	16	58
b_3	17	58	67	28
c_1	17	24	13	28
c_2	13	48	37	24
c_3	37	28	17	48
d_1	13	25	16	24
d_2	36	24	13	45
d_3	16	45	36	25
	choice 1		choice 2	

Figure 20: Pairs of residual points in the E, F and G-lines.

Referring to the quad lines, we see that E, F and G have each occurred with four of the a_i, b_i, c_i and d_i, and so must each occur with 8 more of these points. The points which must still occur with both E and F are a_3, b_2, c_3, d_1, and they must occur in pairs. Now a_3d_1 and b_2c_2 have already occurred. This leaves four pairs for the E, F-lines. A similar situation exists for the E, G-lines, and for the F, G-lines. The possible pairs for the E, F, G-lines are shown in Figure 21.

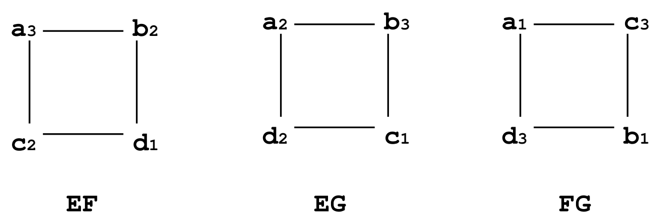

Figure 21: $PP(7)$, pairs of a_i, b_i, c_i, d_i in the E, F, G-lines

At this point we again have to make a choice. Suppose that the pair a_3b_2 occurs in an E-line. Referring to the left side of the table in Figure 20, we see that the pair of residual points common to a_3 and b_2 is 67. It follows that a_3b_267 occur together in an E-line. By Figure 21, c_2d_113 also occurs in an E-line. The F-lines are then forced to contain a_3c_248 and b_2d_125. This in turn forces the remaining pairs in the E, F, G-lines. If we had chosen a_3c_2 instead of a_3b_2 for the E-lines, the distribution would again have been forced. The two possible outcomes for each of the two previous choices are shown in Figure 22. Notice that the only difference between the first and third possibilities is in the residual points $1, 2, \ldots, 8$; similarly for the second and fourth possibilities.

We turn now to the residual lines. The yellow points u, v, w, x, y, z must still occur two more times each in the 12 remaining residual lines. Point u has already occurred with $b_1, b_2, b_3, c_1, c_2, c_3$ in the residual lines – see Figure 17. Point u has occurred with a_3d_1 in the quad lines – see Figure 19. Therefore it must still occur with a_1, a_2, d_2, d_3. But the pairs a_1d_3 and a_2d_2 occur in the E, F, G-lines – see Figure 21. We conclude that u occurs with the pairs a_1d_2 and a_2d_3 in the residual lines. A similar line of reasoning for v, w, x, y, z completely determines the distribution of the a_i, b_i, c_i, d_i in the residual lines – see Figure 23.

We are now close to completing $PP(7)$. We give the names $h_1, \ldots, h_6, k_1, \ldots, k_6$ to the remaining letters in the quad lines, as shown in Figure 24. Consider the letters h_1, k_1 in the first quad line. They must each occur three times in the residual lines. They have both occurred with a_1, b_1 and y in a quad line. There are exactly six residual lines which do not contain these points. Two of the possible six lines contain z, so that one of the

E	$e_1a_3b_2$	67	F	$f_1a_1c_3$	37	G	$g_1a_2d_2$	36
E	$e_2c_2d_1$	13	F	$f_2b_1d_3$	16	G	$g_2b_3c_1$	17
E	$e_3a_2b_3$	58	F	$f_3a_3c_2$	48	G	$g_3a_1d_3$	45
E	$e_4c_1d_2$	24	F	$f_4b_2d_1$	25	G	$g_4b_1c_3$	28

E	$e_1a_3c_2$	48	F	$f_1a_1d_3$	45	G	$g_1a_2b_3$	58
E	$e_2b_2d_1$	25	F	$f_2b_1c_3$	28	G	$g_2c_1d_2$	24
E	$e_3a_2d_2$	36	F	$f_3a_3b_2$	67	G	$g_3a_1c_3$	37
E	$e_4b_3c_1$	17	F	$f_4c_2d_1$	13	G	$g_4b_1d_3$	16

E	$e_1a_3b_2$	58	F	$f_1a_1c_3$	48	G	$g_1a_2d_2$	45
E	$e_2c_2d_1$	24	F	$f_2b_1d_3$	25	G	$g_2b_3c_1$	28
E	$e_3a_2b_3$	67	F	$f_3a_3c_2$	37	G	$g_3a_1d_3$	36
E	$e_4c_1d_2$	13	F	$f_4b_2d_1$	16	G	$g_4b_1c_3$	17

E	$e_1a_3c_2$	37	F	$f_1a_1d_3$	36	G	$g_1a_2b_3$	67
E	$e_2b_2d_1$	16	F	$f_2b_1c_3$	17	G	$g_2c_1d_2$	13
E	$e_3a_2d_2$	45	F	$f_3a_3b_2$	58	G	$g_3a_1c_3$	48
E	$e_4b_3c_1$	28	F	$f_4c_2d_1$	24	G	$g_4b_1d_3$	25

Figure 22: $PP(7)$, 1-factorizations in the E, F, G-lines

z-lines contains h_1 and the other k_1. Since h_1 and k_1 have not been differentiated, we are free to place them as shown in Figure 24. In a similar way, we find that one occurrence of each of $h_2, k_2, \ldots, h_6, k_6$ can be placed uniquely in the residual lines.

Now h_1 and k_1 have occurred with A, B, E, and so must still occur in the F and G-lines. They have both occurred with a_1, b_1. Furthermore, h_1 has occurred with c_1, d_1; and k_1 has occurred with c_2, d_2. Referring to Figure 22, we see that in each possible situation for the F, G-lines, the placement of h_1 and k_1 is forced. Similarly the placement of $h_2, k_2, \ldots, h_6, k_6$ in the E, F, G-lines is forced. The result is shown in Figure 25. Notice in particular that two of the arrangements of the E, F, G-lines are not possible – h_1, k_1 cannot be placed in the G-lines for the second and fourth arrangements. At this point, h_1 has occurred with $a_1, b_1, c_1, d_1, a_3, c_2, a_2, d_2$. It must still occur with b_2, b_3, c_3, d_3. This forces its remaining two occurrences in the residual lines. Similarly for $h_2, k_2, \ldots, h_6, k_6$. The result is shown in Figure 26. Referring to Figure 25, we find that e_1 has occurred with a_3, b_2, h_2, h_5. This leaves only two possible residual lines in which e_1 can appear. Since e_1 must appear twice in the residual lines, its placement there is forced. In a similar way, we find that the placements of $e_1, \ldots, e_4, f_1, \ldots, f_4, g_1, \ldots, g_4$ in the residual lines is forced. This is also shown in Figure 26.

We must still complete the A, B, C, D-lines. Notice that h_1 and k_1

u a1 d2
u a2 d3
v b1 c1
v b3 c3
w a1 c2
w a3 c3
x b1 d1
x b2 d3
y a2 b2
y a3 b3
z c1 d1
z c2 d2

Figure 23: $PP(7)$, the remaining residual lines

ABEy	a1 b1 h1 k1
ACFw	a2 c1 h2 k2
ADGu	a3 d1 h3 k3
BCGv	b2 c2 h4 k4
BDFx	b3 d2 h5 k5
CDEz	c3 d3 h6 k6
EFuv	g1 g2 g3 g4
EGwx	f1 f2 f3 f4
FGyz	e1 e2 e3 e4

u a1 d2 h4
u a2 d3 k4
v b1 c1 h3
v b3 c3 k3
w a1 c2 h5
w a3 c3 k5
x b1 d1 h2
x b2 d3 k2
y a2 b2 h6
y a3 b3 k6
z c1 d1 h1
z c2 d2 k1

Figure 24: $PP(7)$, quad lines and residual lines

must still occur in the C, D-lines. Also, h_1 has already occurred with v, x, y, z. Therefore it must occur with u in the C-lines, and w in the D-lines. Consequently, it appears with points $2, 7$ in the C-lines, and with $1, 5$ in the D-lines. Referring to the third row of Figure 25, we see that h_1 occurs with points $3, 7$ there. Since this is clearly impossible, we conclude that the first row of Figure 25 is the correct arrangement of the E, F, G-lines. This corresponds to the arrangement of the A, B, C, D-lines shown in Figure 19. The placement of the remaining points $h_1, \ldots, h_6, k_1, \ldots, k_6$ is then forced.

There remain the twelve points $e_1, \ldots, e_4, f_1, \ldots, f_4, g_1, \ldots, g_4$. Each must occur once in the A, B, C, D-lines. We find that e_1 has occurred so far with $u, v, y, z, a_1, a_3, b_1, b_2, c_1, d_2, h_2, h_3, h_4, h_5, h_6, k_2, k_5, k_6, f_3, f_4, 6, 7$ (see Figures 24, 25 and 26). This forces it to occur in the second A-line, second B-line, fourth C-line and fourth D-line. Similarly all the remaining twelve points are completely forced. See Figure 27. It is tedious, but easy to verify that each pair of points occurs exactly once in the design. This completes

E	$e_1a_3b_2h_2h_5$	67	F	$f_1a_1c_3h_3k_4$	37	G	$g_1a_2d_2h_1k_6$	36
E	$e_2c_2d_1k_2k_5$	13	F	$f_2b_1d_3k_3h_4$	16	G	$g_2b_3c_1k_1h_6$	17
E	$e_3a_2b_3h_3h_4$	58	F	$f_3a_3c_2h_1h_6$	48	G	$g_3a_1d_3h_2k_5$	45
E	$e_4c_1d_2k_3k_4$	24	F	$f_4b_2d_1k_1k_6$	25	G	$g_4b_1c_3k_2h_5$	28

E	$e_1a_3c_2$	48	F	$f_1a_1d_3$	45	G	$g_1a_2b_3h_1$ *	58
E	$e_2b_2d_1$	25	F	$f_2b_1c_3$	28	G	$g_2c_1d_2k_1$ *	24
E	$e_3a_2d_2$	36	F	$f_3a_3b_2$	67	G	$g_3a_1c_3$	37
E	$e_4b_3c_1$	17	F	$f_4c_2d_1$	13	G	$g_4b_1d_3$	16

E	$e_1a_3b_2h_2h_5$	58	F	$f_1a_1c_3h_3k_4$	48	G	$g_1a_2d_2h_1k_6$	45
E	$e_2c_2d_1k_2k_5$	24	F	$f_2b_1d_3k_3h_4$	25	G	$g_2b_3c_1k_1h_6$	28
E	$e_3a_2b_3h_3h_4$	67	F	$f_3a_3c_2h_1h_6$	37	G	$g_3a_1d_3h_2k_5$	36
E	$e_4c_1d_2k_3k_4$	13	F	$f_4b_2d_1k_1k_6$	16	G	$g_4b_1c_3k_2h_5$	17

E	$e_1a_3c_2$	37	F	$f_1a_1d_3$	36	G	$g_1a_2b_3h_1$ *	67
E	$e_2b_2d_1$	16	F	$f_2b_1c_3$	17	G	$g_2c_1d_2k_1$ *	13
E	$e_3a_2d_2$	45	F	$f_3a_3b_2$	58	G	$g_3a_1c_3$	48
E	$e_4b_3c_1$	28	F	$f_4c_2d_1$	24	G	$g_4b_1d_3$	25

Figure 25: $PP(7)$, the E, F, G-lines

the unique construction of $PP(7)$ from an initial non-Fano quad. We state this as:

Theorem 3 *The projective plane of order 7 is completely determined by any non-Fano quad. It has an automorphism group mapping any quad to any other quad.*

Proof. The initial quad $\{A, B, C, D\}$ was chosen arbitrarily. The completion of the plane was unique.

5 Conclusion

Simple combinatorial proofs of the uniqueness of the planes of orders 3, 4, and 5, and the non-existence of $PP(6)$ have been given by R. Stanton in [7]. Stanton has also recently proved an elegant "Pattern Theorem" for the distribution of points in planes containing a non-Fano quad [8]. The only existing proof of the uniqueness of $PP(8)$ appears to be an exhaustive computer search. In [2], Hall points out the need for further methods to prove the uniqueness of $PP(8)$. The sum of squares theorem may be useful for this.

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u	a_1	d_2	h_4	h_6	k_2	e_1	f_4
u	a_2	d_3	k_4	k_1	h_5	e_2	f_3
v	b_1	c_1	h_3	k_5	k_6	f_3	e_1
v	b_3	c_3	k_3	h_1	h_2	f_4	e_2
w	a_1	c_2	h_5	k_6	k_3	e_3	g_2
w	a_3	c_3	k_5	h_4	k_1	e_4	g_1
x	b_1	d_1	h_2	k_4	h_6	g_1	e_3
x	b_2	d_3	k_2	h_3	h_1	g_2	e_4
y	a_2	b_2	h_6	k_3	k_5	f_1	g_4
y	a_3	b_3	k_6	k_2	k_4	f_2	g_3
z	c_1	d_1	h_1	h_5	h_4	g_3	f_1
z	c_2	d_2	k_1	h_2	h_3	g_4	f_2

Figure 26: $PP(7)$, completion of the residual lines

A	$b_2c_3d_2e_3f_3g_3$		1
A	$b_3c_2d_3e_1f_1g_1$		2
A	$h_5h_6e_4f_2$	v	35
A	$h_4k_6e_2g_4$	x	47
A	$k_5k_4f_4g_2$	z	68

B	$a_3c_1d_3e_3f_4g_4$		3
B	$a_2c_3d_1e_1f_2g_2$		4
B	$h_2k_6e_4f_1$	u	18
B	$h_6h_3e_2g_3$	w	26
B	$k_2k_3f_3g_1$	z	57

C	$a_3b_1d_2e_2f_1g_2$		5
C	$a_1b_3d_1e_4f_3g_4$		6
C	$h_1k_5e_3f_2$	u	27
C	$k_1k_3e_1g_3$	x	38
C	$h_3h_5f_4g_1$	y	14

D	$a_2b_1c_2e_4f_4g_3$		7
D	$a_1b_2c_1e_2f_2g_1$		8
D	$k_1k_2e_3f_1$	v	46
D	$h_1k_4e_1g_4$	w	15
D	$h_2h_4f_3g_2$	y	23

Figure 27: $PP(7)$, completion of the A, B, C, D-lines

References

[1] Robert Bumcrot, *Modern Projective Geometry*, Holt, Rinehart and Winston, Inc., New York, 1969.

[2] Marshall Hall, Jr., "The uniqueness of the projective plane with 57 points", Proc. Amer. Math. Soc. 4 (1953) 912-916.

[3] Marshall Hall, Jr., "Correction to "The uniqueness of the projective plane with 57 points"", Proc. Amer. Math. Soc. 5 (1954) 994-997.

[4] R. Moufang, "Zur Struktur der projektiven Geometrie der Ebene", Math. Ann. 105 (1931), pp. 536-601.

[5] Günter Pickert, *Projektive Ebene*, Springer-Verlag, Berlin, 1955.

[6] William A. Pierce, "The impossibility of Fano's configuration in a projective plane with eight points per line", Proc. Amer. Math. Soc. 4 (1953) 908-912.

[7] R.G. Stanton, "A combinatorial approach to small projective geometries", Bulletin of the ICA 17 (1996), pp. 63-70.

[8] R.G. Stanton, "Four colour classes in projective planes with non-Fano quads", Ars Combinatoria 83 (2007), to appear.